
Trigonometric Identities

— By Samuel Li and Vincent Huang —

Introduction

Trigonometric identities can simplify expressions involving trigonometric functions that would otherwise be impossible to directly compute without a calculator.

For instance, trigonometric identities can allow us to compute the trigonometric functions evaluated at some untraditional angles without a calculator.

There will also be problems that ask you to *prove* a trigonometric identity that you will have not seen before. One has to use trigonometric identities to simplify both sides of the identity you want to prove to arrive at something trivial (like $1=1$).

The purpose of this chapter is to familiarize ourselves with important trigonometric identities and their applications.

Basic Trigonometric Identities

Here are some fundamental conversions that you should be aware of:

- $\sin(-x) = -\sin(x)$
- $\cos(-x) = \cos(x)$
- $\tan(-x) = -\tan(x)$
- $\cot(-x) = -\cot(x)$
- $\csc(-x) = -\csc(x)$
- $\sec(-x) = \sec(x)$
- $\sin(90^\circ - x) = \cos(x)$
- $\tan(90^\circ - x) = \cot(x)$
- $\csc(90^\circ - x) = \sec(x)$
- $\cos(90^\circ - x) = \sin(x)$
- $\cot(90^\circ - x) = \tan(x)$
- $\sec(90^\circ - x) = \csc(x)$

Basic Trigonometric Identities

Here are some fundamental conversions that you should be aware of:

- $\sin(180^\circ - x) = \sin(x)$ and $\csc(180^\circ - x) = \csc(x)$
- $\cos(180^\circ - x) = -\cos(x)$ and $\sec(180^\circ - x) = -\sec(x)$
- $\tan(180^\circ - x) = -\tan(x)$ and $\cot(180^\circ - x) = -\cot(x)$

Examples:

Prove that $\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$

Examples:

Simplify $\frac{\tan x}{\sec x}$.

Pythagorean Identities

Recall this basic identity that follows from the Pythagorean Theorem:

$$\sin^2(x) + \cos^2(x) = 1$$

There are two more corresponding identities that follow immediately from the one mentioned above. Can you see why?

$$1 + \cot^2(x) = \csc^2(x)$$

$$1 + \tan^2(x) = \sec^2(x)$$

Algebraic Simplifications

Sometimes, one needs to be aware of algebraic identities that can help with simplification, for instance with difference of squares.

Example: Simplify $\frac{\sin^2 x}{1 - \cos x}$.

Examples:

Show that $\frac{(\csc x)(\sec x)}{1 + \tan x} = \cot x$.

Examples:

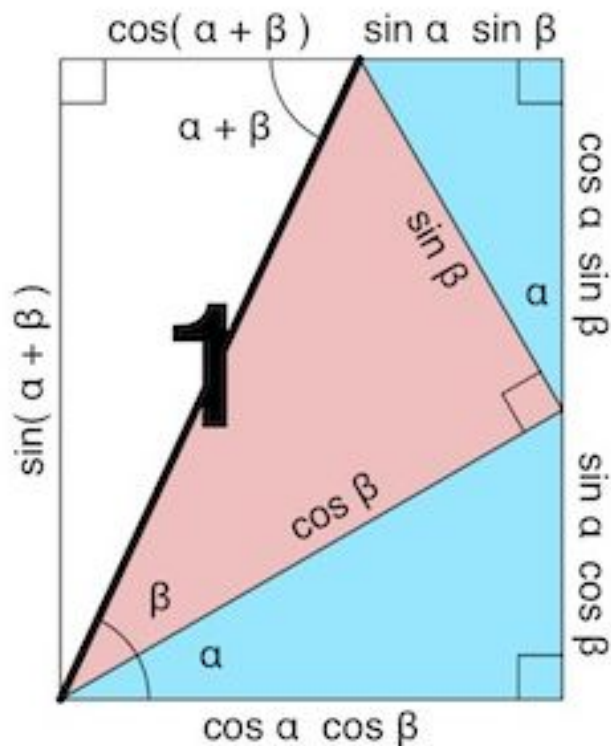
Show that $\frac{1}{1 - \sin x} + \frac{1}{1 + \sin x} = 2 \sec^2 x$

Sum and Differences of Angles

$$\sin(x + y) = \sin(x) \cos(y) + \cos(x) \sin(y)$$

$$\cos(x + y) = \cos(x) \cos(y) - \sin(x) \sin(y)$$

$$\tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x) \tan(y)}$$



Computing Unconventional Angles

Compute $\sin 15^\circ$ using $\sin(45^\circ - 30^\circ)$. What about $\sin 75^\circ$?

What about $\cos 15^\circ$ and $\cos 75^\circ$?

Special Case: Double Angle Identities

By simply setting $x=y$ in the angle sum formulas, we arrive at:

$$\sin(2x) = 2 \sin(x) \cos(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

$$\tan(2x) = \frac{2 \tan(x)}{1 - \tan^2(x)}$$

In particular, there are multiple formulations of the cosine double angle identity that needs to be familiarized:

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2 \sin^2(x) = 2 \cos^2(x) - 1$$

Deriving these identities aren't as hard as recognizing them and using them appropriately.

Examples:

Simplify $\frac{\cos x - \sin x}{\cos x + \sin x}$

Half-Angle Identities

In fact, we can derive half-angle identities as a direct corollary of the double-angle identities:

$$\cos(x) = 1 - 2 \sin^2 \left(\frac{x}{2} \right) \implies \sin \left(\frac{x}{2} \right) = \pm \sqrt{\frac{1 - \cos(x)}{2}}$$

$$\cos(x) = 2 \cos^2 \left(\frac{x}{2} \right) - 1 \implies \cos \left(\frac{x}{2} \right) = \pm \sqrt{\frac{1 + \cos(x)}{2}}$$

The $\pm$ indicates that the sign is either positive or negative depending on the quadrant $x/2$ is in.

Half-Angle Identities

Example: $\cos(15^\circ) = \pm \sqrt{\frac{1 + \cos 30^\circ}{2}}$

$$= \pm \sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}}$$

$$= \pm \sqrt{\frac{2 + \sqrt{3}}{4}}$$

In this case, 15° is in quadrant I, and because cosine is positive in quadrant I, the $\pm$ must be positive.

Recall that using sum angle identity, $\cos 15^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}$. Why would

$$\sqrt{\frac{2 + \sqrt{3}}{4}} = \frac{\sqrt{6} + \sqrt{2}}{4}?$$

Tangent Half-Angle:

$$\tan \frac{x}{2} = \frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}.$$

This can be derived from the sine and cosine double identities, although the proof is slightly less direct and requires casework on the quadrant that x resides in.

Examples:

If $\sin \theta = \frac{1}{4}$, what is $\cos 2\theta$?

Examples:

Compute $\cos(20^\circ) \cos(40^\circ) \cos(80^\circ)$.

Examples (Triple Angle identities):

Show that $\sin(3x) = 3 \sin x - 4 \sin^3 x$ and $\cos(3x) = 4 \cos^3 x - 3 \cos x$.
(Hint: Use $\sin(2x + x)$)

Product-to-sum Identities:

$$\sin(x) \sin(y) = \frac{1}{2}(\cos(x - y) - \cos(x + y))$$

$$\sin(x) \cos(y) = \frac{1}{2}(\sin(x - y) + \sin(x + y))$$

$$\cos(x) \cos(y) = \frac{1}{2}(\cos(x - y) + \cos(x + y))$$

Sum-to-Product Identities:

$$\sin(x) + \sin(y) = 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

$$\sin(x) - \sin(y) = 2 \sin\left(\frac{x-y}{2}\right) \cos\left(\frac{x+y}{2}\right)$$

$$\cos(x) + \cos(y) = 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

$$\cos(x) - \cos(y) = -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$$

Examples:

Compute $\frac{\sin 14^\circ + \sin 46^\circ}{\cos 16^\circ}$.

Review of All Trig Identities:

In this chapter, we explored the following identities:

$$\begin{array}{llll}\sin^2 \theta + \cos^2 \theta = 1 & \sin(-\theta) = -\sin \theta & \sin(90^\circ - \theta) = \cos \theta & \sin(180^\circ - \theta) = \sin \theta \\ \tan^2 \theta + 1 = \sec^2 \theta & \cos(-\theta) = \cos \theta & \cos(90^\circ - \theta) = \sin \theta & \cos(180^\circ - \theta) = -\cos \theta \\ \cot^2 \theta + 1 = \csc^2 \theta & \tan(-\theta) = -\tan \theta & \tan(90^\circ - \theta) = \cot \theta & \tan(180^\circ - \theta) = -\tan \theta\end{array}$$

Angle sum and difference identities:

$$\begin{array}{ll}\sin(\alpha + \beta) = \sin \alpha \cos \beta + \sin \beta \cos \alpha & \sin(\alpha - \beta) = \sin \alpha \cos \beta - \sin \beta \cos \alpha \\ \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta & \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \\ \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} & \tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}\end{array}$$

Double angle and half-angle identities:

$$\begin{array}{ll}\sin 2x = 2 \sin x \cos x & \cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}} \\ \cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x & \sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}} \\ \tan 2x = \frac{2 \tan x}{1 - \tan^2 x} & \tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta} = \frac{1 - \cos \theta}{\sin \theta}\end{array}$$

Product-to-sum and sum-to-product identities:

$$\begin{array}{ll}\cos \alpha \cos \beta = \frac{1}{2} (\cos(\alpha + \beta) + \cos(\alpha - \beta)) & \cos x + \cos y = 2 \cos \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right) \\ \sin \alpha \sin \beta = \frac{1}{2} (\cos(\alpha - \beta) - \cos(\alpha + \beta)) & \cos x - \cos y = -2 \sin \left(\frac{x+y}{2} \right) \sin \left(\frac{x-y}{2} \right) \\ \sin \alpha \cos \beta = \frac{1}{2} (\sin(\alpha + \beta) + \sin(\alpha - \beta)) & \sin x + \sin y = 2 \sin \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right) \\ & \sin x - \sin y = 2 \sin \left(\frac{x-y}{2} \right) \cos \left(\frac{x+y}{2} \right)\end{array}$$

Solving Trig Equations

Trigonometric equations are quite self-explanatory: You have an equation with a lot of trigonometry parts. In order to solve trig equations, you should use trig identities and whatnot to simplify both sides of the equation to solve it.

For most problems, you want to get all simplify both sides to contain trig functions with the same angle, because otherwise, it is would be very difficult to solve.

Keep in mind that trig equations, unless otherwise specified, may have infinitely many solutions. It is also possible that a trig equation has no solutions, for example,

Examples:

Solve over all x : $\sin(2x) = \frac{1}{2}$

Examples:

Solve over all x : $\sin^2 \left(\pi x + \frac{\pi}{4} \right) = 0$

Examples:

Solve over all x : $\sin^2(x) - \cos^2(x) = \frac{3}{2}$

Examples:

Solve for all $x \in [0, 2\pi)$: $2 \cos^2 x - \cos x - 1 = 0$

Examples:

Solve for all $x \in [0, 2\pi)$: $2 \sin^2(x) + 9 \cos(x) = 2 \cos(2x) + 7$

Examples:

Solve over all x : $\sin x + \sin(3x) = 0$

Geometric Trig

Trig in triangles now

Sorry for hand drawn images

Introduction

Trigonometry appears in triangles as well. Although they are mainly in right triangles, there are also applications in other triangles, including the Law of Cosines and Law of Sines.

Many properties of trigonometry can be proved using triangles. In geometry, trigonometry is also used for some solutions (bash out side lengths, angles, etc.)

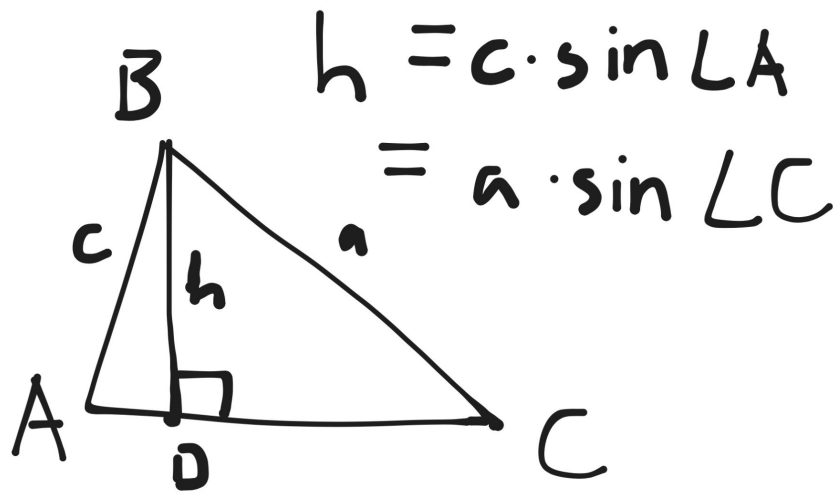
The purpose of this chapter is to introduce how trigonometry is used inside geometry, primarily in triangles.

Right Triangle Trig

Sin is opposite over hypotenuse, Cos is adjacent over hypotenuse, Tan is opposite over adjacent, other trig functions are based off these three. You can also find the height of the triangle.

From this you can also get identities like $\cos^2 + \sin^2 = 1$, also the height of the triangle is shown to the right

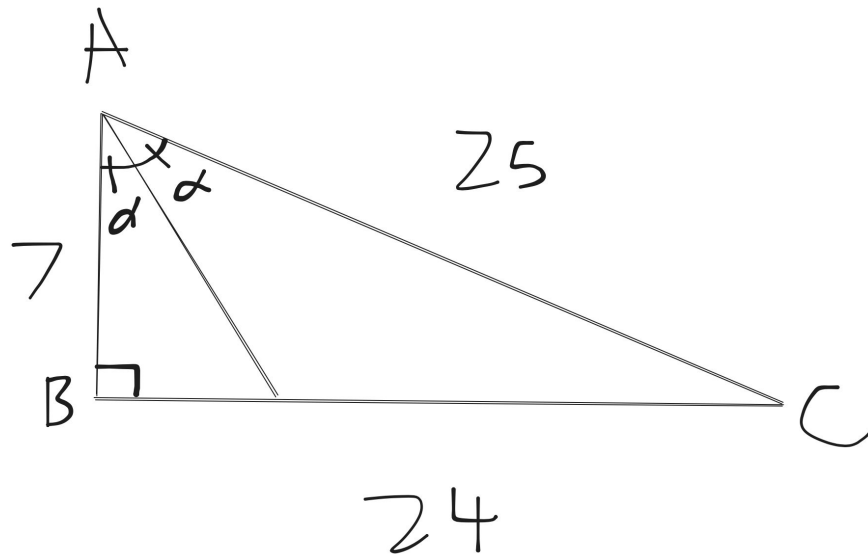
(it's messy because i drew it with a trackpad sorry)



Examples:

In the following diagram:

Find $\sin \alpha$.



Example:

In triangle ABC, $AB = 4$, $AC = 7$, angle $BAC = 60$ degrees. What is the area of triangle ABC?

Law of Sines

In a triangle, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

R is circumradius, a, b, c are respective sides of triangle.

Can you prove the above formula?

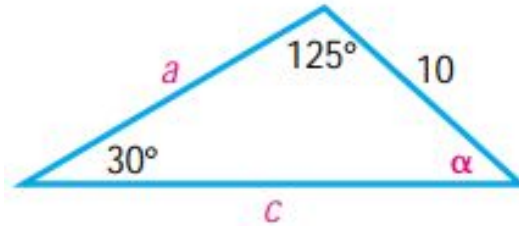
(Hint: for 2R, draw the circumcenter and connect it to the vertices)

Example:

In triangle ABC, $AB = 5$, angle $ACB = 30$, angle $ABC = 45$, what is AC ?

Example:

Solve the following triangle (that is, find all unknown sides and angles):



(you can use a calculator)

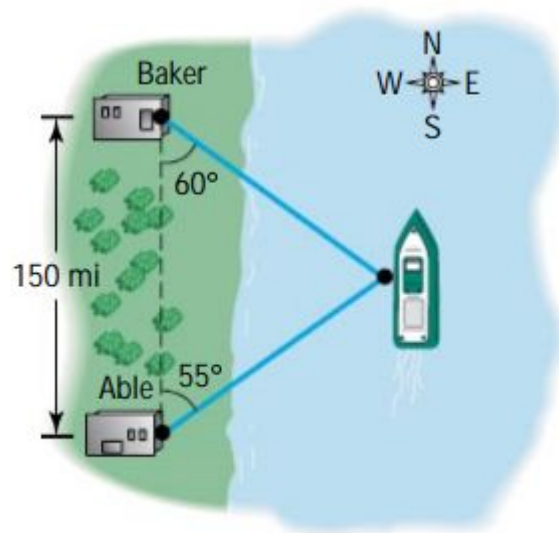
Example:

In triangle PQR, $PQ = 5$ and angle $PRQ = 60$, what is the circumradius of PQR?

Example: (Application)

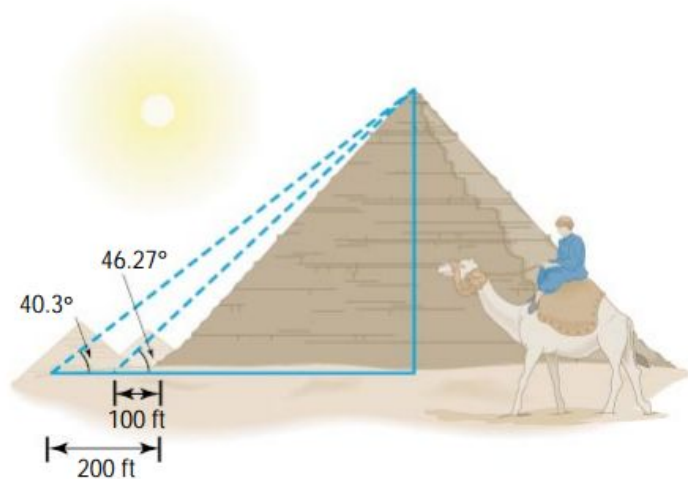
Coast Guard Station Able is located 150 miles due south of Station Baker. A ship at sea sends an SOS call that is received by each station. The call to Station Able indicates that the ship is located $N55^\circ E$; the call to Station Baker indicates that the ship is located $S60^\circ E$.

- (a) How far is each station from the ship?
- (b) If a helicopter capable of flying 200 miles per hour is dispatched from the station nearest to the ship, how long will it take to reach the ship?



Example: (Application)

Using the information provided in the diagram, find the current height of the Great Pyramid.



Law of Cosines

With two sides and their included angle, you can find the length of the last

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Can you prove this? (Hint: Use the Pythagorean Theorem)

Example:

In triangle ABC, $AB = 7$, $BC = 9$, and $\angle ABC = 30^\circ$, what is AC^2 ?

Example:

In $\triangle ABC$, $AB = 8$, $AC = 15$, $BC = 13$, what is angle BAC ?

Example:

Two boats move away from each other at 6 mph and 11 mph, respectively. In the beginning, they were faced 60 degrees away from each other. After 2 hours, how far apart are the boats?

Heron's formula

Another area formula.. A bit complicated though

Let s be semiperimeter : $\frac{a + b + c}{2}$

Heron's formula states that the area of ABC is $\sqrt{s(s-a)(s-b)(s-c)}$

This is used when you have all the side lengths and no angles.

Example:

In $\triangle ABC$, $AB = 13$, $AC = 14$, $BC = 15$. What is the area of $\triangle ABC$?

Example:

In $\triangle ABC$, $AB = 4$, $AC = 9$, $BC = 6$, what is the area of $\triangle ABC$?