

Week 3 Class 1

Review of trig application to geo + polar coordinate
system

Quick Review of Concepts:

For a triangle ABC with $BC = a$, $AC = b$, $[ABC] = \frac{1}{2}ab \sin(\angle ACB)$

Law of Cosines:

For a triangle ABC with $BC = a$, $AC = b$, $AB = c$, we have that

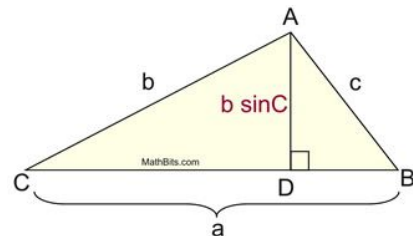
$$c^2 = a^2 + b^2 - 2ab \cos(\angle C)$$

Law of Sines:

For a triangle ABC with $BC = a$, $AC = b$, $AB = c$, we have that

$$\frac{a}{\sin(\angle A)} = \frac{b}{\sin(\angle B)} = \frac{c}{\sin(\angle C)} = 2R$$

where R is the circumradius of triangle ABC .



Quick Review of Concepts:

Heron's Formula

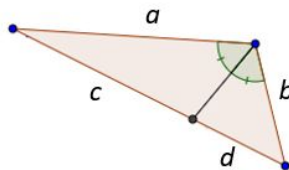
$$Area = \sqrt{s(s - a)(s - b)(s - c)}$$

where s is the semiperimeter (half of $a+b+c$).

Angle Bisector Theorem

Angle Bisector Theorem

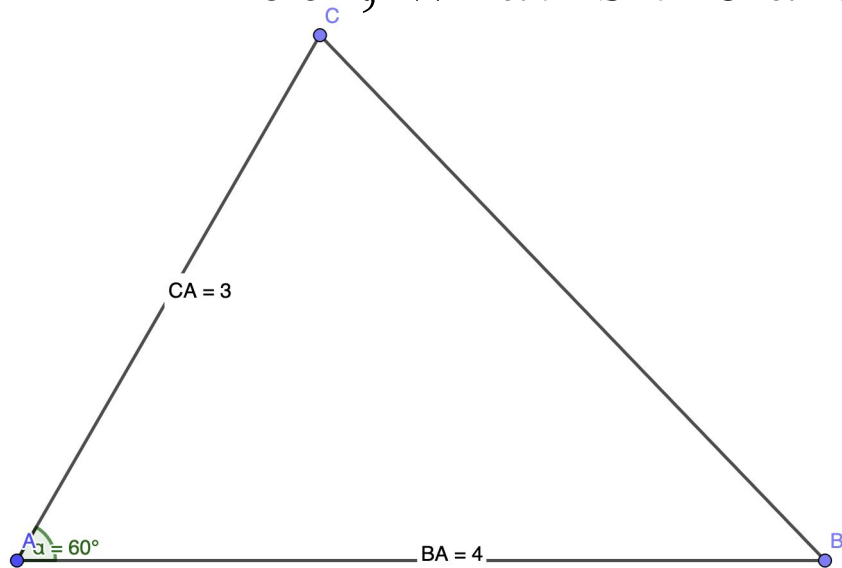
If a ray bisects an angle of a triangle, then it divides the opposite side of the triangle into segments that are proportional to the other two sides.



$$\frac{a}{b} = \frac{c}{d}$$

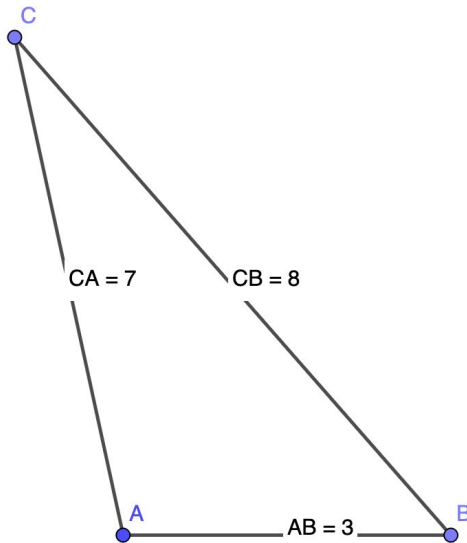
Review Practice Problems

Given that a triangle ABC has $AB = 4$, $AC = 3$, and $\angle A = 60^\circ$, what is the area of triangle ABC ?



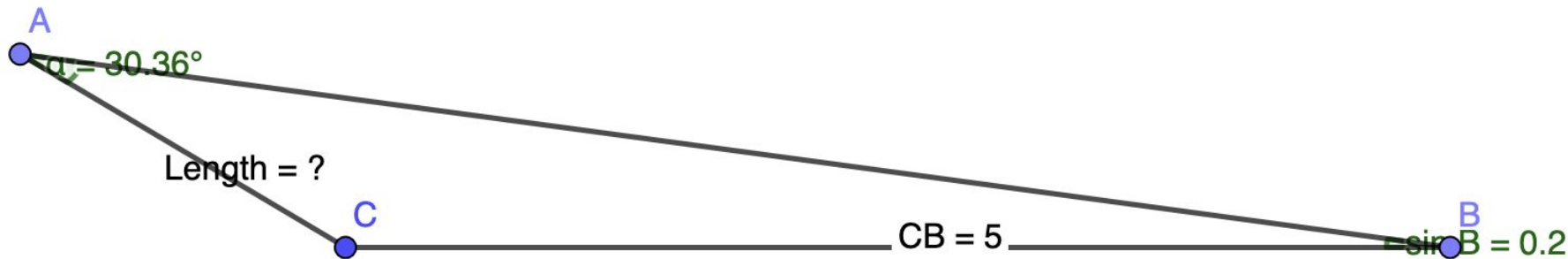
Review Practice Problems

Given that a triangle ABC has $AB = 3$, $AC = 7$, $BC = 8$, find $\angle B$.



Review Practice Problems

Given that a triangle ABC has $BC = 5$, $\angle A = 30^\circ$, $\sin \angle B = 0.2$, find the length of CA .



Review Practice Problems

If a triangle has side lengths 10, 17, 21, then what is the area of the triangle?

Hint: What formula do you want to use here?

Review Practice Problems

The perimeter of a triangle is 12 times the average of the sines of the angles of the triangle. If one side of the triangle has length 2, what are the possible measures of the angles opposite that side?

Hint: First write out the equation and compare it to law of sines.

New concept: Polar Coordinate System

Polar Coordinates are of the form (r, θ) where the r -coordinate is the distance from the point to the origin and the θ -coordinate is the **angular coordinate** which is the clockwise angle from the x-axis to the line connecting the origin and the specified point.

Quick practice:

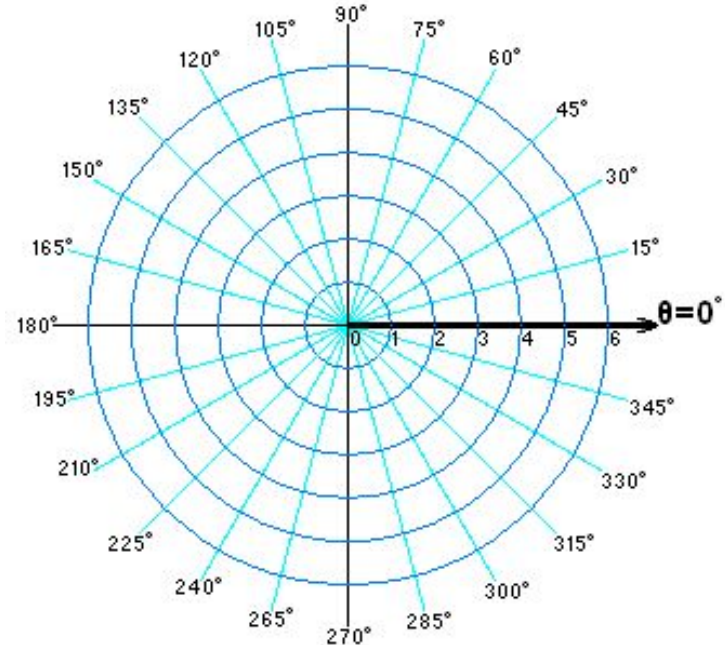
Locate where the following polar points are:

$(2, 0^\circ)$

$(1, 30^\circ)$

$(5, 75^\circ)$

[just draw them out on the graph on the right using zoom annotation if you can]



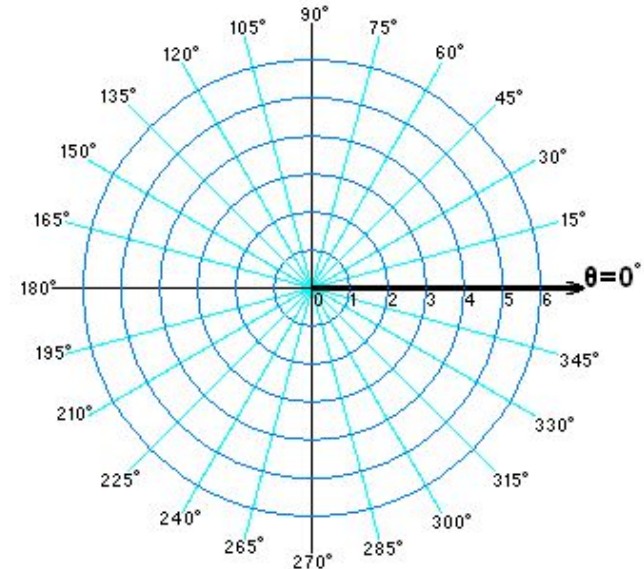
New concept: Polar Coordinate System

Polar Coordinates are of the form (r, θ) where the r -coordinate is the distance from the point to the origin and the θ -coordinate is the **angular coordinate** which is the clockwise angle from the x-axis to the line connecting the origin and the specified point.

Thus, (r, θ) corresponds to the rectangular coordinate:
 $(r \cos \theta, r \sin \theta)$

Proof of why this is true:

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New concept: Polar Coordinate System

Polar Coordinates are of the form (r, θ) where the r -coordinate is the distance from the point to the origin and the θ -coordinate is the **angular coordinate** which is the clockwise angle from the x-axis to the line connecting the origin and the specified point.

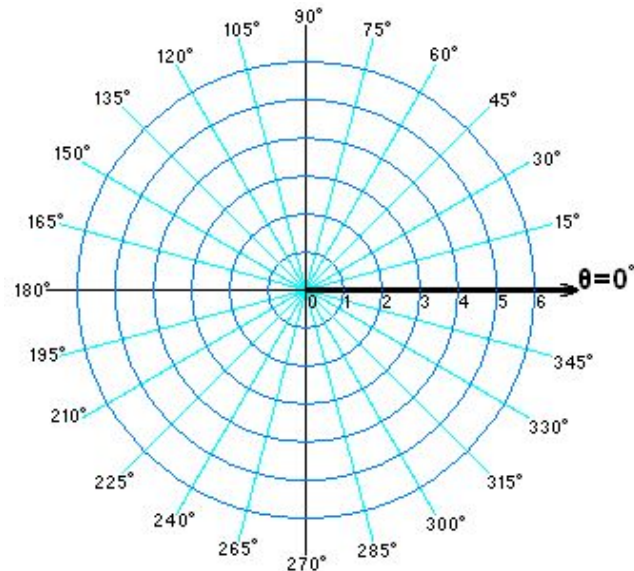
Thus, (r, θ) corresponds to the rectangular coordinate:

$$(r \cos \theta, r \sin \theta)$$

Ex: $(2, 45^\circ)$ is $(\sqrt{2}, \sqrt{2})$

$(2, 30^\circ)$ is $(\sqrt{3}, 1)$

Also note that $(r, \theta) = (-r, \theta + 180^\circ)$ since its a half-circle turn. Try proving this using the above formula.



Polar Coordinate System Practice

Convert each of the following polar coordinates to rectangular coordinates.

1. $(8, 60^\circ)$
2. $(12, -120^\circ)$

Polar Coordinate System Practice

Convert the following rectangular coordinates to polar coordinates.

1. $(0, 2)$
2. $(3, \sqrt{3})$

As a general rule, you first want to find the radius r which can be calculated by taking the square root of the sum of squares of the x and y coordinates.

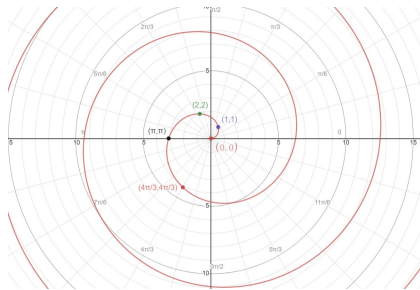
Polar Coordinate System Practice

1. What are the rectangular coordinates of the point with polar coordinates (r, θ) ?
2. What is the graph of $\theta = 20^\circ$?
3. Find all ways to represent the point $(0, 1)$ in polar coordinates.

Polar Equations

Polar equations are another type of parametric equation with the angle θ as the parameter.

For example, $r = \theta$.



The distance to the origin r is a function of the angle.

Ex: $r=3$ is the equation of a circle with radius 3 centered at the origin.

Ex: $r = 1/\cos \theta$ is the vertical line $x = 1$.

Transforming Polar Equations

Example:

$$r = 4 \sin \theta$$

Note that $x^2 + y^2 = r^2 = r \times r = 4r \sin \theta$.

Since $y = r \sin \theta$, thus $x^2 + y^2 = 4y$.

This is now the equation of a circle.

We then complete the square to obtain the standard form of the equation.

$$x^2 + y^2 - 4y = 0$$

$$x^2 + (y^2 - 4y + 4) = 4$$

$$x^2 + (y - 2)^2 = 4$$

Hence, it is a circle centered at $(0, 2)$ with radius 2.

Transforming Polar Equations

Transform this into rectangular coordinates:

$$r = \cos \theta$$

Transforming Polar Equations

Transform this into rectangular coordinates:

$$r = \frac{4}{1 - \cos\theta}$$

Transforming Polar Equations

Transform this into rectangular coordinates:

$$r = \sin \theta - \cos \theta$$

Transforming Polar Equations

Transform this into rectangular coordinates:

$$r^2 = \cos \theta$$

Transforming Polar Equations

Transform this into rectangular coordinates:

$$r = \frac{3}{3 - \cos\theta}$$

Transforming Rectangular to Polar

Transforming rectangular to polar is simpler.

In any equation $y = f(x)$, replace y with $r \sin \theta$ and x with $r \cos \theta$. Then simplify the equation.

For example: (tbh, this is a more trivial example)

$$x^2 + y^2 = 1$$

$$\Rightarrow (r \cos \theta)^2 + (r \sin \theta)^2 = 1$$

$$\Rightarrow r^2 (\cos^2 \theta + \sin^2 \theta) = 1$$

$$\Rightarrow r^2 = 1$$

$$\Rightarrow r = 1$$

Transforming Rectangular Equations

Practice: Turn the following into polar form.

$$(x - 3)^2 + y^2 = 9$$

$$2xy = 1$$

Transforming Rectangular Equations

Practice: Turn the following into polar form.

$$4x^2y = 1$$

$$y = -3$$

Transforming Rectangular Equations

Practice: Turn the following into polar form.

$$y^2 = 2x$$

Week 3 Class 1 Homework

Precalculus Enhanced with Graphing Utilities

Chapter 8.1, Page 580

- Problems 11-18, 31-37, 55-65

Bonus:

AOPS Precalculus Textbook Page 189 Problem 5.31

Week 3 Class 2

Parameterization, and polar equations
3d coordinates and equation of planes

Parameterization - polar and rectangular

Parameterization - instead of a single equation that relates x and y , we can write both x and y in terms of a third variable.

For instance, to model the path of a particle in the 2d plane over time, we could write x and y as functions of a variable t which represents time.

Ex: This is a system of parametric equations which models linear motion.

$$x = 2 - 3t$$

$$y = 1 + 2t$$

To transform this parameterized equation into standard form, we first write t in terms of x and substitute that into the second equation to write y in terms of x .

Parameterization Basics Practice. Write it as a relation of x and y .

1. $x = \sin t, y = \cos t.$

2. $x = 1 - t, y = 2t^2 - t$

3. $x = 3 + t, y = 10 - t$

4. $x = 12 - t, y = 13 + 5t$

Hint: $\sin^2 x + \cos^2 x = 1$

Parameterization Basics Practice

Use parametric equations to model the following situation:

An ant starts from the point $(0, 1)$ and moves with a rightward velocity of 5 m/s and a downward velocity of 3 m/s . Model this with parametric equations of x and y in terms of the time t .

Parameterization Basics Practice

Find the graph of this parametric equation:

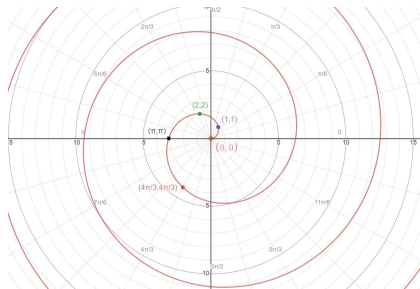
$$x = 3 \cos 2t$$

$$y = -3 \sin 2t$$

Re: Polar Equations

Polar equations are another type of parametric equation with the angle *theta* (θ) as the parameter.

For example, $r = \theta$.



The distance to the origin r is a function of the angle.

Ex: $r=3$ is the equation of a circle with radius 3 centered at the origin.

Ex: $r = 1/\cos \theta$ is the vertical line $x = 1$.

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Polar Equations

Convert the following polar equations into parametric equations of x and y in terms of θ :

1. $r = 2 \sin \theta$

2. $r = \cos \theta$

Polar Equations

Convert the following polar equation into parametric equations of x and y in terms of: θ

$$r = \frac{2}{1 + \cos \theta}$$

Polar Equations

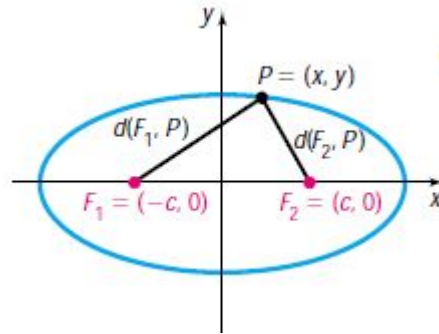
Convert the following polar equations into parametric equations of x and y in terms of θ :

$$r = 3 + 2 \sin \theta$$

$$r = \frac{1}{\sin \theta} . \text{ What does the graph of this one look like?}$$

Ellipses

An ellipse is a conic section, meaning that it is the intersection of a tilted plane, and a cone. Another definition is the locus of all points where the sum of the distances between that point and the two foci are a constant, as shown below.



The equation of an ellipse is $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$. Its polar form is $r = \frac{ed}{1 + e \cos(\theta)}$.

Practice With Ellipses

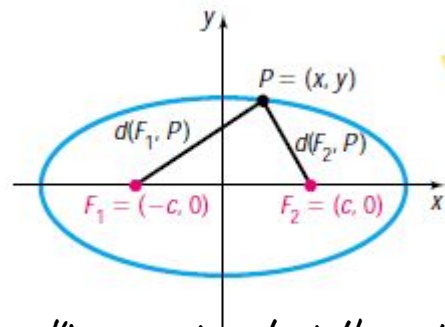
The equation of an ellipse is $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$.

Based on this information, try to derive a few small results:

1. Where is the ellipse centered at?
2. Where are its left and rightmost points? What about its bottom and uppermost points?

More about Ellipses

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$



An ellipse centered at the origin

The center is (h, k) . The length of the horizontal and vertical axes are $2a$ and $2b$ respectively.

(The horizontal axis goes from $(h - a, k)$ to $(h + a, k)$. The vertical axis goes from $(h, k - b)$ to $(h, k + b)$.)

Practice problem:

Find the semimajor length of $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

More about Ellipses

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Definitions:

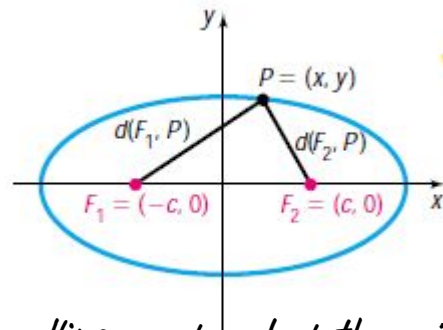
The major axis is the longer axis, and the shorter axis is called the minor axis.

The semimajor length is $\frac{1}{2}$ of the major axis. Semiminor length is $\frac{1}{2}$ of the minor axis.

The foci fall on the major axis. For an ellipse centered at the origin, the foci are at the following positions:

The foci are:

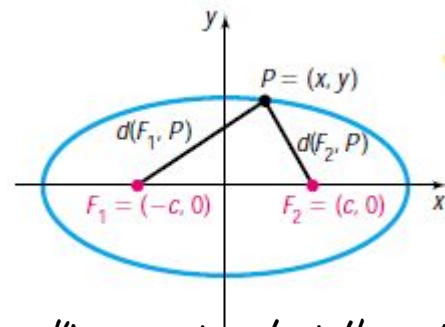
$$\begin{cases} (-\sqrt{a^2 - b^2}, 0) \text{ and } (\sqrt{a^2 - b^2}, 0), & \text{if } a > b \\ (0, -\sqrt{b^2 - a^2}) \text{ and } (0, \sqrt{b^2 - a^2}), & \text{if } a < b \end{cases}$$



An ellipse centered at the origin

More about Ellipses

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$



An ellipse centered at the origin

The foci are:

$$\begin{cases} (-\sqrt{a^2 - b^2}, 0) \text{ and } (\sqrt{a^2 - b^2}, 0), & \text{if } a > b \\ (0, -\sqrt{b^2 - a^2}) \text{ and } (0, \sqrt{b^2 - a^2}), & \text{if } a < b \end{cases}$$

Derivation of the Foci of the Ellipse

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More about Ellipses

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

The foci are:

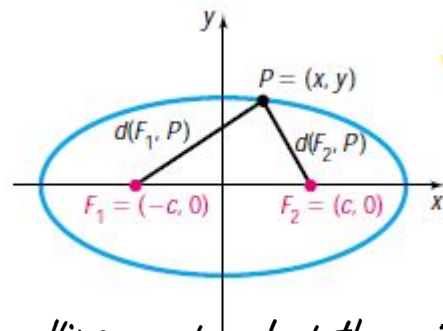
$$\begin{cases} (-\sqrt{a^2 - b^2}, 0) \text{ and } (\sqrt{a^2 - b^2}, 0), & \text{if } a > b \\ (0, -\sqrt{b^2 - a^2}) \text{ and } (0, \sqrt{b^2 - a^2}), & \text{if } a < b \end{cases}$$

The foci are always a distance of $\sqrt{|a^2 - b^2|}$ from the center of the ellipse.

The axis they are on is dependent on whichever one is longer.

Practice Problem:

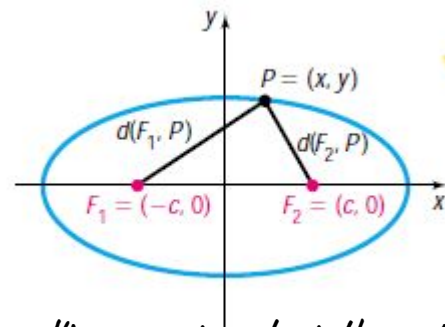
Find the foci of $\frac{x^2}{4} + \frac{y^2}{9} = 1$.



An ellipse centered at the origin

More about Ellipses

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$



An ellipse centered at the origin

As mentioned earlier, one definition of the Ellipse is the locus of all points P such that

$PF_1 + PF_2$ is a constant value.

Here, we will show how to derive the equation of an ellipse from the two foci:

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Ellipse Exercises

Find the coordinates of the foci for the ellipse given by:

$$9x^2 + 25y^2 = 225.$$

Ellipse Exercises

Problem:

Determine the center and foci of the ellipse given by the equation:

$$\frac{(x - 2)^2}{16} + \frac{(y + 3)^2}{25} = 1$$

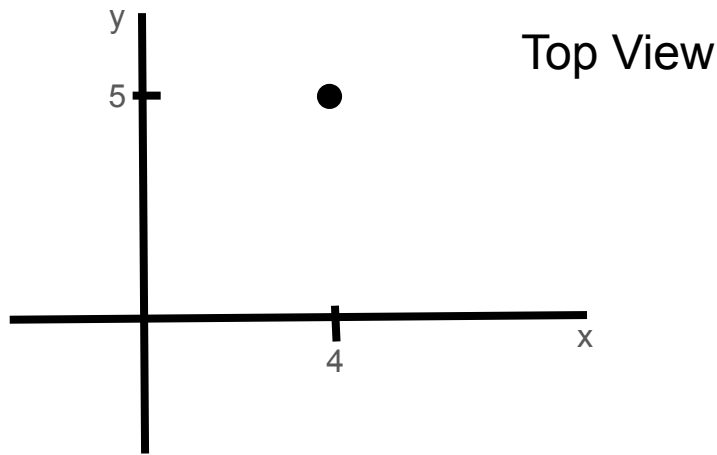
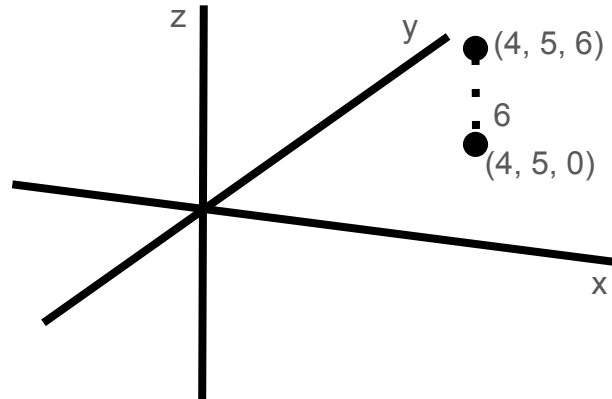
Ellipse Exercises

Write the standard equation of the ellipse centered at the origin with a focus at $(0, 3)$ and a vertex at $(0, 5)$.

3D Coordinates - Introduction

Two coordinates (x, y) can identify a point's location in the two-dimensional Cartesian plane. Similarly, three coordinates (x, y, z) can identify a point's location in the three-dimensional Cartesian plane.

Example: $(4, 5, 6)$ denotes a point that is 6 units above the xy - plane at the point $(4, 5)$.



3D Coordinates Practice Problems

1. If a point lies on the xy -plane, what can you say about its z -coordinate?
2. Is it possible for two different points in space to have the same x and y coordinates?
3. Given that a point is a distance of 4 units from the xy -plane, find all possible values of its z -coordinate.

3D Coordinates - Distance Formula

By considering opposite vertices of a rectangular prism with each edge parallel to a coordinate axis, one can derive a formula for the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in the 3D plane (similar to the 2D version):

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

3D Coordinates - Distance Formula (Proof)

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

● (x_2, y_2, z_2)

(x_1, y_1, z_1) ●

Distance Formula Practice Problems (1)

1. Let point A be located at $(-3, 4, 6)$ and let point B be located at $(4, -1, 2)$. Find AB .
2. If point P is located at $(-3, 4, 12)$ and point Q is located at $(6, -8, -8)$, which point is closer to the origin?

Distance Formula Practice Problems (2)

1. Given that the coordinates of point R are $(-4, 2, z)$ and that R is at most 14 units away from the origin, what is the maximum possible value of z ?
2. Consider the equation $(x - 3)^2 + (y - 2)^2 + (z + 4)^2 = 36$. When all points in the 3D plane satisfying this equation are plotted, what figure is formed?

Hint: Try to manipulate the expression into something familiar.

3D Coordinates - Equation of a Plane (1)

A **plane** is a flat, two-dimensional surface extending infinitely far in all directions.

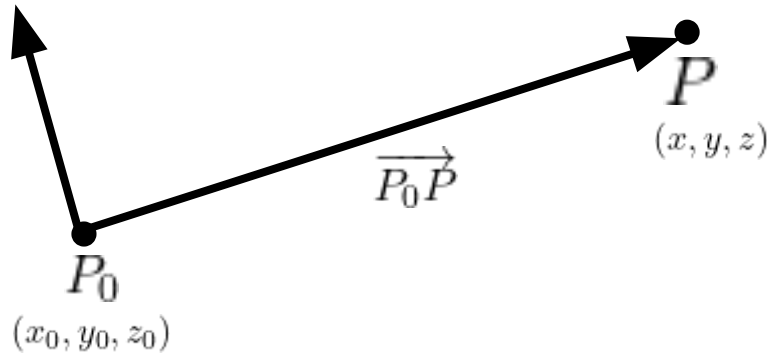
The general equation of a plane can be written as

$$Ax + By + Cz + D = 0$$

for some constants A, B, C, D .

Most natural proofs of the plane equation use a **normal vector**, which is a vector perpendicular to the given plane. Since vectors have not been taught yet, we just explain the main idea of the proof.

3D Coordinates - Equation of a Plane (2)



Equation of a Plane Practice Problems

1. Find the equations of the xy , yz , and xz -planes. Check that they match with the general plane equation.
2. Find the equation of the plane passing through the points $(4, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 1)$.
3. How many non-collinear points are needed to determine a unique plane?

Week 3 Class 2 Homework

Precalculus Enhanced with Graphing Utilities

Chapter 8.2, Page 598

- Problems 37-45, 67, 68, 69, 70, 82, 83, 85, 88

Chapter 8.6, Page 637-638

- Problems 15-26, 67, 71, 72

Bonus:

AOPS Precalculus Textbook Page 187, Questions 5.3.8 and 5.3.9