

Complex Numbers

Catherine J.

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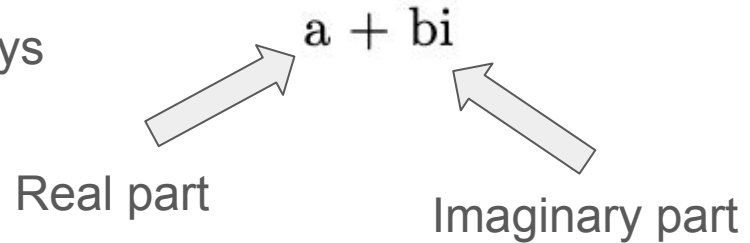
Definition

Real Numbers

- Numbers that can be plotted on a number line
- Includes all rational and irrational numbers
- The square of a real number will always be nonnegative

Complex Numbers

- $i = \sqrt{-1}$
- Complex numbers are of the form $a + bi$ where a and b are real



Complex Number Arithmetic

Addition/Subtraction:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

Multiplication:

$$\begin{aligned} & (a + bi)(c + di) \\ &= a(c + di) + bi(c + di) \\ &= ac + adi + bci + bdi^2 \\ &= ac + adi + bci - bd \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

Examples

1. Find all values of z such that $z^2 + 2025 = 0$
2. Evaluate i^{2026} .
3. Find all values of k such that the product $(3+2i)(3+ki)$ is a real number.

Conjugates

If $z = a + bi$, then $a - bi$ is called the conjugate of z , denoted $\bar{z}$.

A nonreal number and its conjugate are referred to as a conjugate pair.

$z\bar{z}$ is a real number.

$$\overline{z + w} = \bar{z} + \bar{w}$$

$$\overline{zw} = \bar{z} \cdot \bar{w}$$

$$\operatorname{Re}(z) = \frac{z + \bar{z}}{2} \text{ and } \operatorname{Im}(z) = \frac{z - \bar{z}}{2i}$$

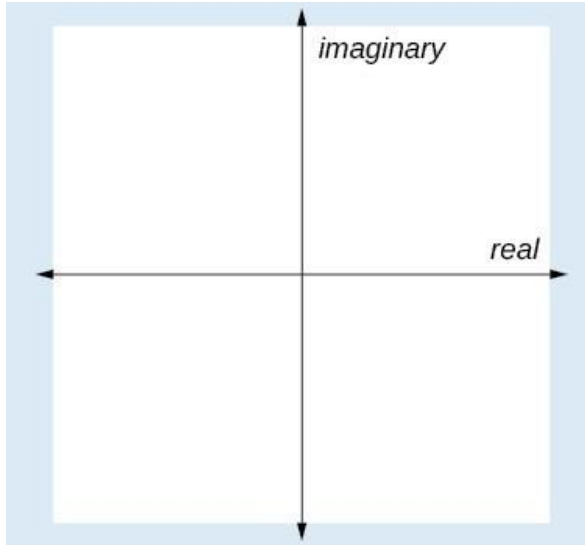
Examples

Write $\frac{2-i}{3+2i}$ as a complex number.

Find all complex numbers such that $\frac{3z-5i}{2z+3} = 3 - 2i$.

The Complex Plane

The magnitude of z , $|z|$ is the distance from z to the origin on the plane.



If $z = a + bi$, then $|z| = \sqrt{a^2 + b^2}$.

$$z\bar{z} = |z|^2$$

Example

Find the complex number z such that the magnitude of z is 5 and $z + \bar{z} = 6$.

Review

The Fundamental Theorem of Algebra: every nonconstant one-variable polynomial with complex coefficients has at least one complex root $\Rightarrow$ every polynomial of degree n has exactly n complex roots

The Factor Theorem: every complex number a , the binomial $(x - a)$ is a factor of the polynomial $f(x)$ if and only if $f(a) = 0$.

The Rational Root Theorem: if $\frac{p}{q}$ is a rational root of

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where p , q , and all of a_i are integers and $\gcd(p, q) = 1$, then a_0 is a multiple of p and a_n is a multiple of q .

Complex Roots

$$x^2 + 19x + 88 = (x + 8)(x + 11) \Rightarrow x = -8, x = -11$$

$$x^2 + 4 \longleftarrow \text{Can't be factored into terms with real coefficient, no real roots}$$

$$x^2 + 4 = (x - 2i)(x + 2i) \Rightarrow x = 2i, x = -2i$$

Examples

1. Find all roots of $g(x) = 2x^3 + 5x^2 + 15x + 18$.
2. Find all roots of $f(x) = 2x^4 - 23x^2 + 27t - 36$.

Conjugate Pairs

Let $p(x)$ be a polynomial with real coefficients. If z is a root of $p(x)$, then $\bar{z}$ is also a root of $p(x)$. In other words, the nonreal roots of $p(x)$ come in conjugate pairs.

Homework

Book: AoPS Precalculus

Exercises:

6.2.3 - 6.2.7

6.3.4 - 6.3.7

Book: Precalculus Enhanced with Graphing Utilities

page 237, 3.7 #1-40 1 mod 4

Trigonometry and Complex Numbers

Derek Hu

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- IV. Selected Problems

Section I

Polar Form of Complex Numbers

Polar Form: $r(\cos \theta + i \sin \theta)$

Instead of $a+bi$ or (a, b) like in rectangular coordinates, we can describe a complex number by its distance r and angle θ :

$$z = r(\cos(\theta) + i \sin(\theta)) = (r, \theta)$$

Finding r and θ from $a + bi$:

$$r = \sqrt{a^2 + b^2}$$

$$\theta = \arctan(b/a)$$

Make sure that you have the correct quadrant when finding theta using arctan

Finding a and b from r and θ :

$$a = r \cos(\theta)$$

$$b = r \sin(\theta)$$

Examples: Rectangular $\leftrightarrow$ Polar

Rectangular $\rightarrow$ Polar

$$z = 3 + 6i \implies r = \sqrt{3^2 + 6^2} \quad \theta = \arctan(6/3)$$

$$\implies r = 3\sqrt{5} \quad \theta = 1.107$$

$$\implies z = 3\sqrt{5}(\cos(1.107) + i \sin(1.107))$$

Polar $\rightarrow$ Rectangular

$$z = 2(\cos(\pi/3) + i \sin(\pi/3))$$

$$\implies a = 2 \cos(\pi/3) \quad b = 2 \sin(\pi/3)$$

$$\implies a = 1 \quad b = \sqrt{3}$$

$$\implies z = 1 + i\sqrt{3}$$

Quadrant Warning for Arctan

$$Q1(a > 0, b > 0) : \theta = \arctan(b/a)$$

$$Q2(a < 0, b > 0) : \theta = \pi + \arctan(b/a)$$

$$Q3(a < 0, b < 0) : \theta = -\pi + \arctan(b/a) \quad Q4(a > 0, b < 0) : \theta = \arctan(b/a)$$

Multiplying & Dividing in Polar Form

Polar form makes multiplication and division easy:

$$z_1 \cdot z_2 = r_1 \cdot r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2))$$

Example:

$$2(\cos(\pi) + i \sin(\pi)) \cdot 3(\cos(\pi/2) + i \sin(\pi/2)) = 6(\cos(3\pi/2) + i \sin(3\pi/2))$$

De Moivre's Theorem

Applying the multiplication rule repeatedly gives a powerful result:

$$(r(\cos(\theta) + i \sin(\theta)))^n = r^n(\cos(n\theta) + i \sin(n\theta))$$

Works for all integers (and real) n

Example: Compute $(1+i)^8$

$$\begin{aligned}(1 + i)^8 &= (\sqrt{2}(\cos(\pi/4) + i \sin(\pi/4)))^8 \\ &= (\sqrt{2})^8(\cos(2\pi) + i \sin(2\pi)) \\ &= 16(1 + 0i) = 16\end{aligned}$$

Examples

If $z = 3(\cos 20^\circ + i \sin 20^\circ)$ and $w = 5(\cos 100^\circ + i \sin 100^\circ)$, find the following (leave your answers in polar form):

(a) zw (b) $\frac{z}{w}$

Write $[2(\cos 20^\circ + i \sin 20^\circ)]^3$ in the standard form $a + bi$.

$$\left[\sqrt{3} \left(\cos \frac{5\pi}{18} + i \sin \frac{5\pi}{18} \right) \right]^6$$

Section II

Exponential Form of Complex Numbers

Euler's Formula

One of the most beautiful results in all of mathematics:

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

- e is Euler's number ≈ 2.71828 (base of natural logarithm)
- The complex exponential $e^{i\theta}$ is a point on the unit circle at angle θ
- It connects exponential and trigonometric functions beautifully

Euler's Identity:
$$e^{i\pi} + 1 = 0$$

Exponential Form: $z = re^{i\theta}$

Using Euler's formula, any complex number can be written:

$$z = re^{i\theta} \text{ where } r = |z| \text{ and } \theta = \arg(z)$$

All three forms are useful:

Form	Expression	Best used for
Rectangular	$a + bi$	Addition, subtraction
Polar	$r(\cos \theta + i \sin \theta)$	Multiplication, division
Exponential	$re^{i\theta}$	Powers, roots

Example: $z = 3 - 4i = 5(\cos(-0.927) + i \sin(-0.927)) = 5e^{-0.927i}$

Operations in Exponential Form

Multiplication	$z_1 \cdot z_2$	$r_1 r_2 e^{i(\theta_1 + \theta_2)}$
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Division	$\frac{z_1}{z_2}$	$\frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$
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Exponentiation	z^n	$r^n e^{in\theta}$
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Conjugation	$\bar{z}$	$r e^{-i\theta}$
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Examples

Find $6e^{i\pi/3}$ in rectangular form.

For what positive integers n is $(-3 + i\sqrt{3})^n$ a real number?

Evaluate $(-2\sqrt{3} + 6i)^8$.

Express $-32i$ in exponential form.

Section III

Roots of Unity

What are Roots of Unity?

An n -th root of unity is a complex number z such that $z^n = 1$

$$1 = e^{i \cdot 0} = e^{i \cdot 2\pi k} \text{ for } k \in \mathbb{Z}$$

$$z^n = 1 = e^{in\theta} \implies e^{in\theta} = e^{i \cdot 2\pi k}$$

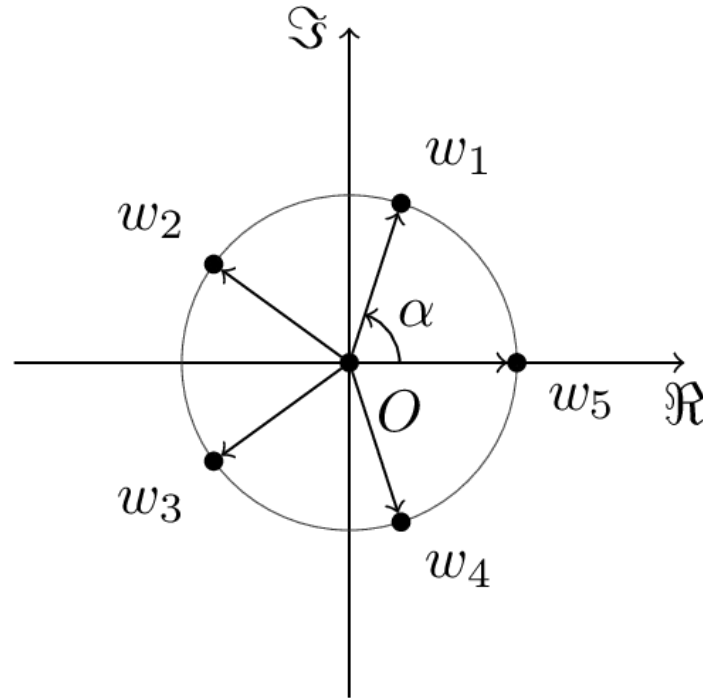
$$\implies in\theta = i \cdot 2\pi k \implies n\theta = 2\pi k \implies \theta = \frac{2\pi k}{n}$$

$$\omega_k = e^{\frac{2\pi i k}{n}} = \cos \frac{2\pi k}{n} + i \sin \frac{2\pi k}{n} \quad k = 0, 1, 2, \dots, n-1$$

These n points are equally spaced on the unit circle!

Roots of Unity on the Unit Circle

All n -th roots of unity lie on the unit circle, evenly spaced by $2\pi/n$ radians:



Cube Roots of Unity ($n = 3$)

Solve $z^3 = 1$.

$$\omega_k = e^{\frac{2\pi i k}{3}} = \cos \frac{2\pi k}{3} + \sin \frac{2\pi k}{3} \quad k = 0, 1, 2$$

Exponential Form: $z = e^0, e^{\frac{2i\pi}{3}}, e^{\frac{4i\pi}{3}}$

Rectangular Form: $z = 1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

Primitive Roots of Unity

An n -th root of unity z is called a primitive n -th root of unity if $z^n = 1$ and $z^k \neq 1$ for $k = 1, 2, \dots, n-1$. For example, i is a primitive 4th root of unity since no $i^k = 1$ for $k = 1, 2, 3$; however i is not a primitive 8th root since $i^4 = 1$ and $4 < 8$.

It is simple to find primitive roots of unity:

Let $\omega = e^{2\pi ki/n}$. ω is a primitive n -th root of unity if and only if $\gcd(k, n) = 1$.

Key Properties of Roots of Unity

All n roots lie on the unit circle

$$|\omega| = 1$$

Equally spaced by angle $2\pi/n$

$$\arg(\omega_{k+1}) - \arg(\omega_k) = \frac{2\pi}{n}$$

Sum of all n -th roots = 0

$$\sum_{k=0}^{n-1} \omega_k = 0$$

$$\prod_{k=0}^{n-1} \omega_k = (-1)^{n+1}$$

Product of all n -th roots = $(-1)^{n+1}$

If ω is a primitive n -th root, all roots are $1, \omega, \omega^2, \dots, \omega^{n-1}$

Conjugate of $\omega_k = \omega_{n-k}$

$$\overline{\omega_k} = \omega_{n-k}$$

Examples

Find all solutions to $x^8 = 1$.

What are the primitive 12th roots of unity?

Let $\omega = e^{4i\pi/7}$. Evaluate $(2 + \omega)(2 + \omega^2)(2 + \omega^3)(2 + \omega^4)(2 + \omega^5)(2 + \omega^6)$.

Example

Find the roots of $g(z) = z^6 + z^4 + z^2 + 1$.

Solution

$$\begin{aligned}g(z) &= z^6 + z^4 + z^2 + 1 \\ \implies z^2 g(z) &= z^8 + z^6 + z^4 + z^2 \\ \implies (z^2 - 1)g(z) &= z^8 - 1 \\ \implies g(z) &= \frac{z^8 - 1}{z^2 - 1}\end{aligned}$$

Thus, the roots of g are the 8th roots of unity excluding ± 1 .

Section IV

Selected Problems

Problems

Find all solutions to $z^3 = -4\sqrt{2} + 4\sqrt{2}i$ in exponential form.

Determine all integer values of θ with $0^\circ \leq \theta \leq 90^\circ$ for which $(\cos \theta + i \sin \theta)^{75}$ is a real number.

Express $\sin 5\theta$ in terms of $\sin \theta$.

Problems

If $x = \frac{-1+i\sqrt{3}}{2}$ and $y = \frac{-1-i\sqrt{3}}{2}$, then find all nonnegative integers n such that $x^n + y^n = -1$.

Let $\theta = \arctan 2$. What quadrant does the angle 8θ lie in?

Challenge Problems (very hard)

Given that z is a complex number such that $z + \frac{1}{z} = 2 \cos 3^\circ$, find the least integer that is greater than $z^{2000} + \frac{1}{z^{2000}}$.

Find a cubic polynomial whose roots are $\cos \frac{2\pi}{7}$, $\cos \frac{4\pi}{7}$, and $\cos \frac{6\pi}{7}$.

Let v and w be distinct, randomly selected roots of the equation $z^{1997} - 1 = 0$. Find the probability that $\sqrt{2\sqrt{3}} \leq |v + w|$.

Homework

Book: Precalculus Enhanced with Graphing Utilities
page 606, 8.3 #11-31 odds, 33, 35, 41, 43, 45, 53-59 odds