

Week 5: Vectors

Day 1

SECTIONS

AoPS Vectors in two dimensions (p.297-p.333)

Chapter 9: Vector addition and scalar multiplication, zero vector, norm of a vector, dot product of vectors, Cauchy Schwarz is derived from dot product, linearly dependency and independence of vectors, projection of a vector.

Troy homework textbook (p.608 - 628) (PDF pages: 695-716)

Trigonometry and Complex Numbers Review

Find the values of these trigonometric ratios, if defined (provide an exact answer):

$\sin(30 \text{ degrees})$

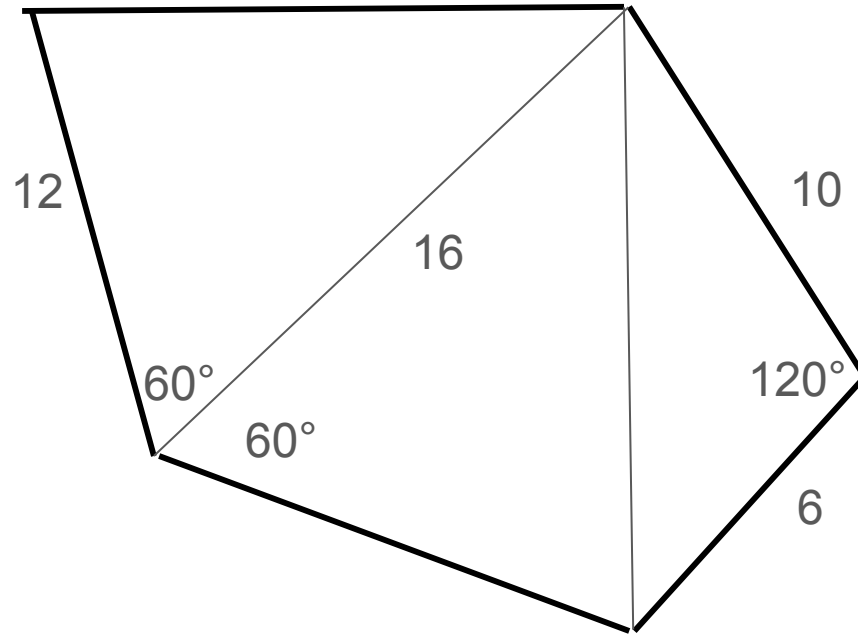
$\cos(315 \text{ degrees})$

$\tan(112.5 \text{ degrees})$

$\cos(363636363615 \text{ degrees})$

Law of Sines and Cosines

Find all possible areas of the pentagon from the given information.



Trigonometry and Complex Numbers Review Part 2

- (a) Find $\tan(\arcsin(1)+\arctan(2)+\arccos(3/5))$
- (b) Find $(2+\sqrt{3}i)/(3+\sqrt{2}i)$ in the form $a+bi$ where a and b are real numbers
- (c) Write $14+34i$ in polar form (to 3 decimal places, angle in degrees)
- (d) Find $(\sqrt{3}/2+i/2)^{2026}$

?

(a) Prove that for all $|z|=1$, $(z-1)/(z+1)$ is purely imaginary.

(i) Hint: Let $z=a+bi$, $a^2+b^2=1$

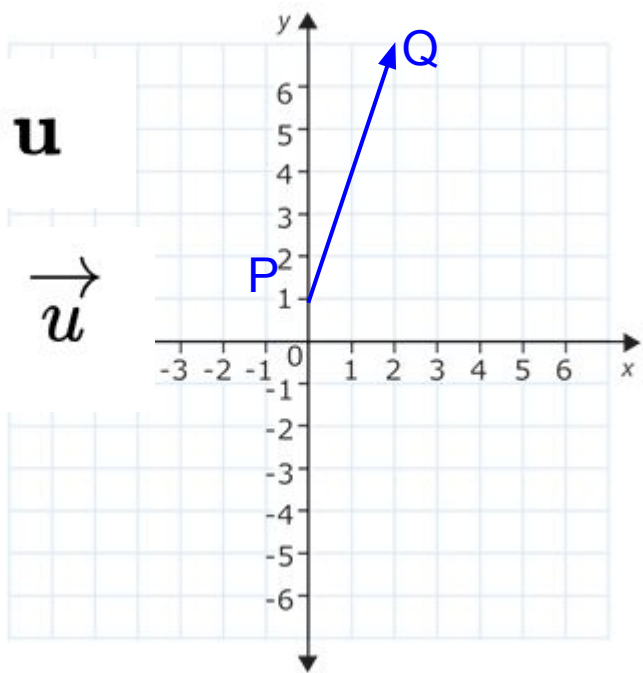
1. $\frac{1}{2}$
2. $a=(6+\sqrt{6})/11$, $b=(3\sqrt{3}-2\sqrt{2})/11$
3. $36.770 \text{ cis}(67.620 \text{ degrees})$
4. $\frac{1}{2}-\sqrt{3}/2 i$
5. Let $z=a+bi$, $a^2+b^2=1$. $(z-1)/(z+1)=(a-1+bi)(a+1-bi)/((a+1)^2+b^2)$. The numerator is $a^2-(1-bi)^2=a^2-1+b^2+2bi=2bi$ which is imaginary, while the denominator is real. So $(z-1)/(z+1)$ is purely imaginary.

Areas: $103\sqrt{3}$, $87\sqrt{3}$

Definitions

A vector is a quantity that has both magnitude and direction, often expressed as $\begin{pmatrix} a \\ b \end{pmatrix}$

or $\langle a, b \rangle$



In a vector from P to Q, we call P the initial point and Q the terminal point. This vector is denoted as $\overrightarrow{PQ}$.

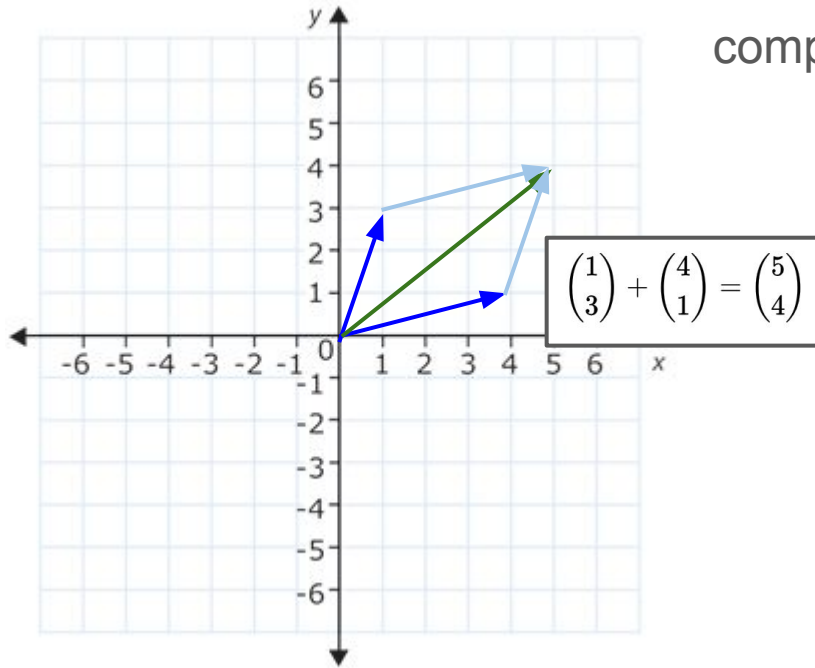
A vector can also be written in a bold face font or with an arrow like **u** or $\vec{u}$.

The zero vector, often denoted as **0** or $\vec{0}$, is the vector with no magnitude.

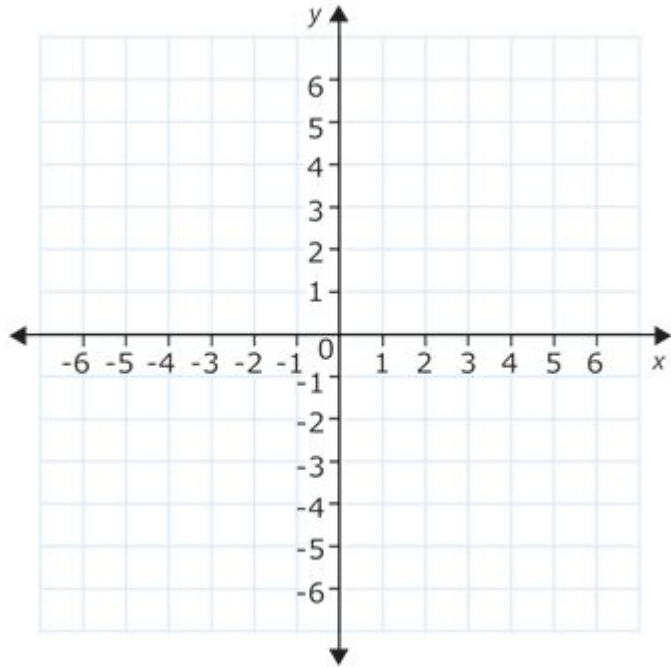
Two vectors are equal if and only if their magnitudes and directions are equal.

Vector Addition

We can add vectors by adding the corresponding components.



Vectors introduction



Let $\mathbf{u} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -5 \\ -2 \end{pmatrix}$.

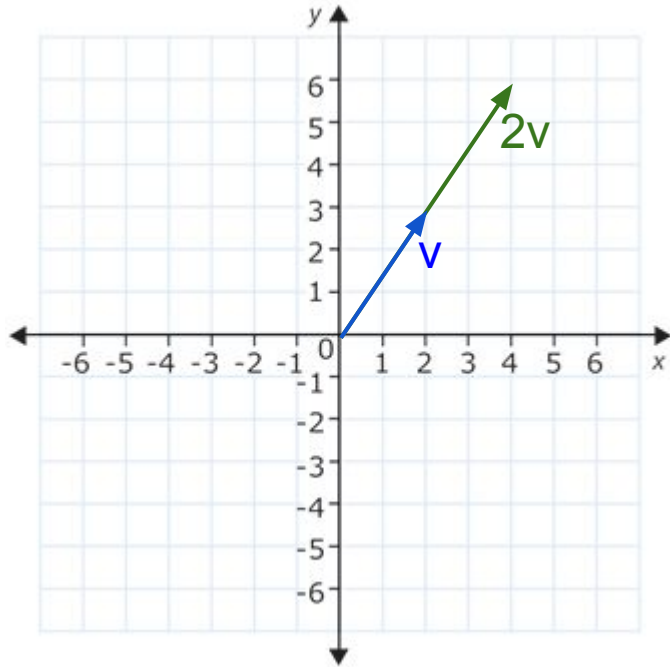
- Plot the vectors $\mathbf{u}$ and $\mathbf{v}$ on the coordinate plane.
- Find $\mathbf{u} + \mathbf{v}$, and plot it on the coordinate plane
- Graph $\mathbf{u}$ from the endpoint of $\mathbf{v}$. Where is the endpoint of this new $\mathbf{u}$?

Horizontal and Vertical components

There are two unit vectors that have special names. The vector $\langle 1, 0 \rangle$ is called **i** and the vector $\langle 0, 1 \rangle$ is called **j**.

Every vector can be written as $a\mathbf{i} + b\mathbf{j}$ for some choice of reals a and b . a is called the horizontal component and b is called the vertical component.

Vector Operations



We can multiply a vector by a constant by multiplying each component by that constant.

This constant can be referred to as a scalar.

For a scalar k and a vector $\begin{pmatrix} a \\ b \end{pmatrix}$, we have the following equation:

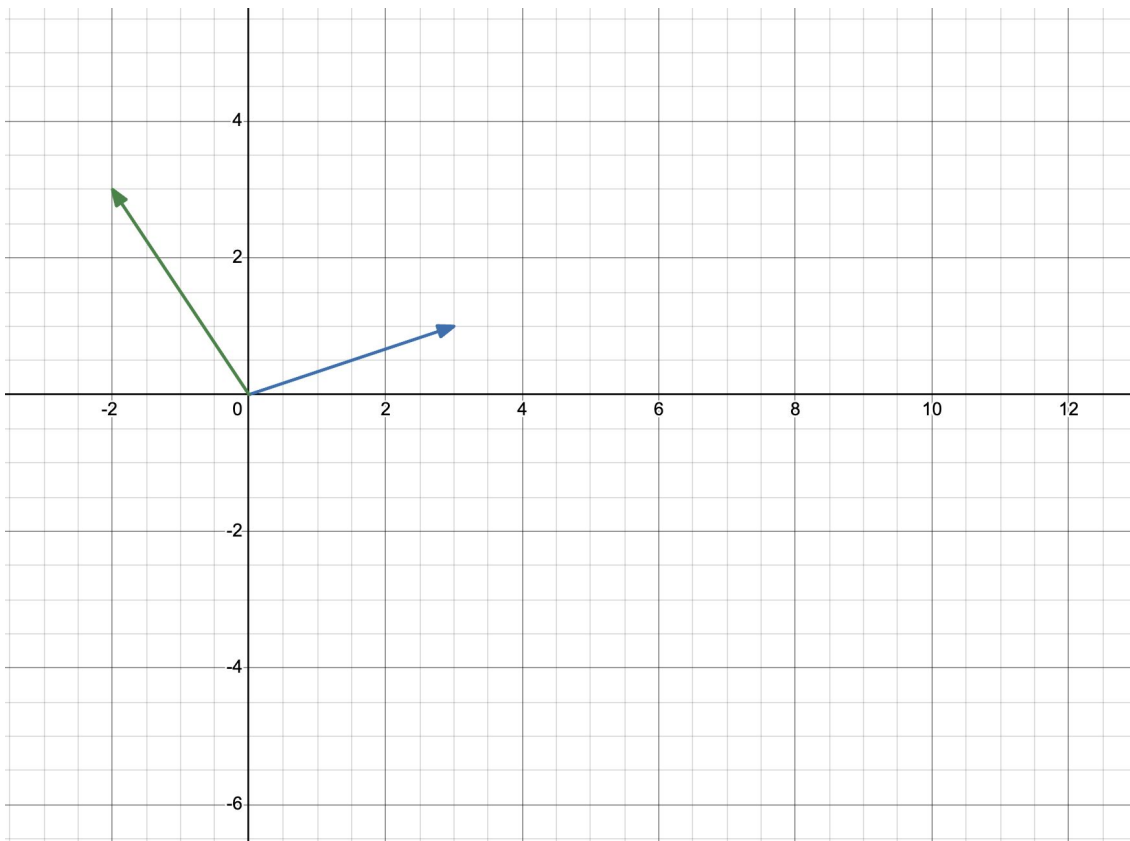
$$k \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ka \\ kb \end{pmatrix}$$

Properties of Vector Operations

Here are some basic properties of vector operations:

1. Vector addition is commutative. $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$
2. Vector addition is associative. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{0}$ is the additive identity: $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$.
4. Basic multiplication rules: $0\mathbf{v} = \mathbf{0}$, $1\mathbf{v} = \mathbf{v}$, $-1\mathbf{v} = -\mathbf{v}$
5. Associativity of scalar multiplication: $a(b\mathbf{v}) = (ab)\mathbf{v}$
6. Distributive properties: $(a+b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$, $a(\mathbf{v}+\mathbf{w}) = a\mathbf{v} + a\mathbf{w}$

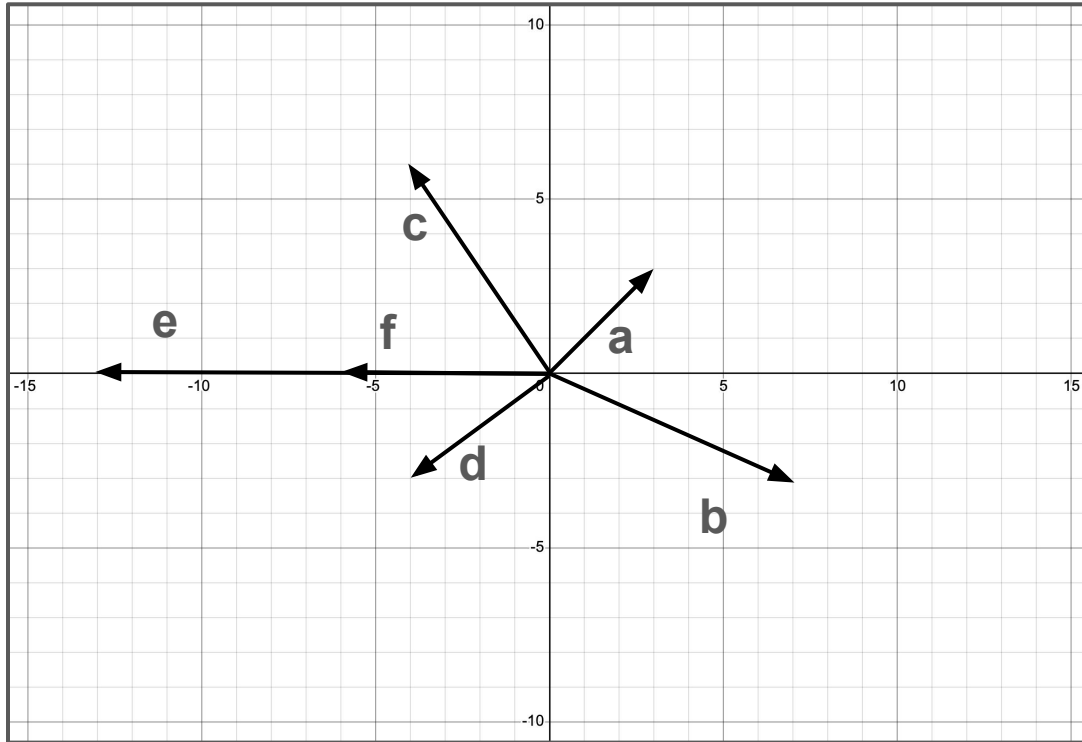
Example Problem 1



Let the blue vector be $\mathbf{u}$, and the green vector be $\mathbf{v}$.

- A. Use coordinate notation for the vectors.
- B. Graph $\frac{1}{2}\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{v}$, $3\mathbf{u} - \mathbf{v}$
- C. Find a vector $\mathbf{w}$ such that
$$\mathbf{v} + \mathbf{w} = \mathbf{u}$$
- D. If $\mathbf{x} = 4\mathbf{u} + 7\mathbf{v}$ and its initial point is $(8, -18)$, find its terminal point.

True or False Part 1



Determine if each of the following statements are true or false.

- A. $\mathbf{b = a - c}$
- B. $\mathbf{d + a}$ has equal length as $\mathbf{e - 2f}$
- C. $\mathbf{A + b + c + d = 0}$
- D. $\mathbf{e + a + d = f/2}$

True or False Part 2

Determine if the following is true or false.

$$a+c=b;$$

$$a-d+h=b;$$

$$c=k;$$

$$d+c+h=0;$$

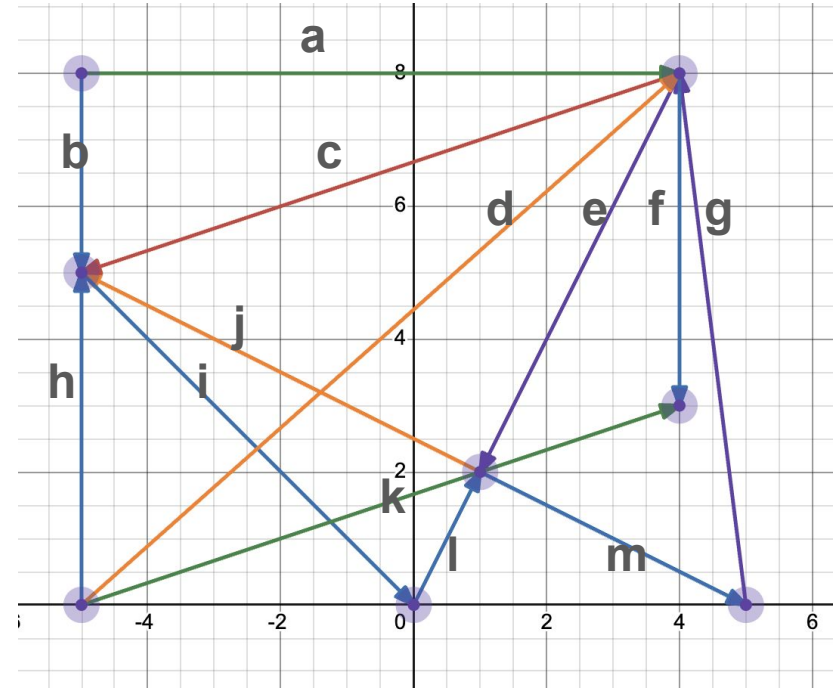
$$e+2l=0;$$

$$j+i+l=0;$$

$$g+c-h+\frac{2}{3}k+m=0;$$

$$k-f=a+b-h;$$

**Colors are used only for visual purposes.*

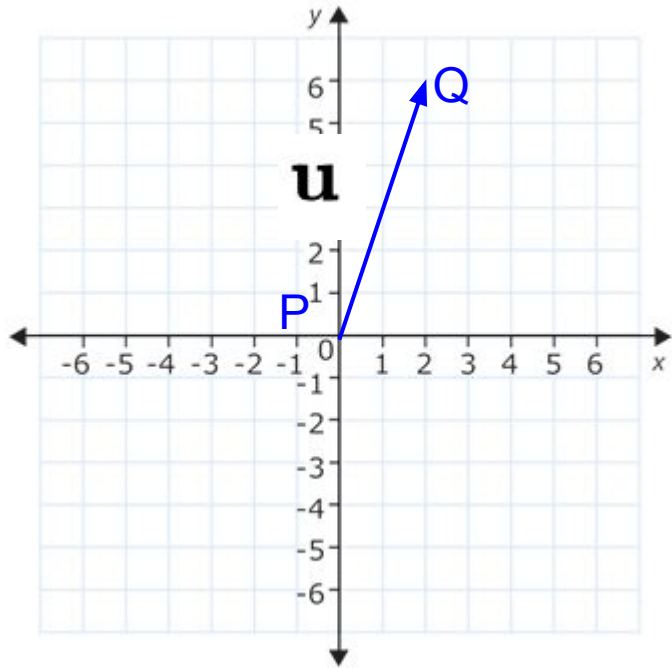


True or False Answers

Part 1: TTFF

Part 2: TTFFFTTF

Direction and Magnitude



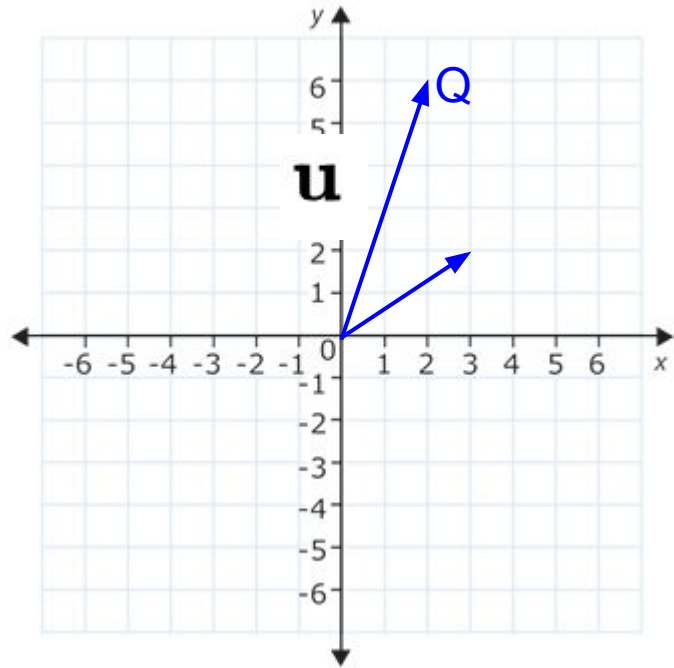
The norm or magnitude of a vector is typically denoted with double absolute value bars ($\|\mathbf{u}\|$). It refers to the distance from the origin to the point (x,y) for the vector $\begin{pmatrix} x \\ y \end{pmatrix}$

A unit vector is a vector with magnitude 1.

It is calculated with the distance formula by using $\|\mathbf{u}\| = \sqrt{x^2 + y^2}$

Example: Find the norm of vector $\overrightarrow{PQ}$ in the image to the left.

The dot product



The dot product of two vectors $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is defined as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$$

Properties of the dot product

Commutative property: $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

Associative property: $\mathbf{u} \cdot (a\mathbf{v}) = a(\mathbf{u} \cdot \mathbf{v})$

Distributive property: $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

Norm property: $\|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u}$

Example Problems 2

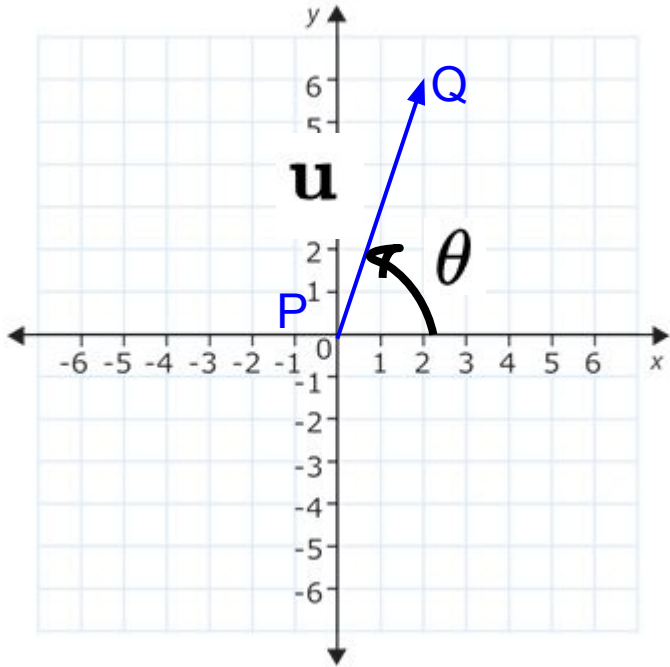
1. Find $\langle 1, 4 \rangle \cdot \langle 3, 4 \rangle$.
2. Let $\mathbf{a} = \langle 1, 4 \rangle$ and $\mathbf{b} = \langle 1, 2 \rangle$. Find all real numbers x such that $\|\mathbf{a} + x\mathbf{b}\| = 13$.
3. Let $\mathbf{a} = \langle 5, 1 \rangle$ and $\mathbf{b} = \langle x, x^2 \rangle$. Find all real numbers x such that $\mathbf{a} \cdot \mathbf{b} = -6$.
4. Let $\mathbf{a} = \langle 6, 7 \rangle$. Find $\mathbf{a} \cdot \mathbf{a}$ and $\|\mathbf{a}\|$.
5. True or False: For any three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$.
6. Find one possible value of $\mathbf{b}$ if $\mathbf{a} = \langle 1, 1 \rangle$, $\mathbf{a} \cdot \mathbf{b} = 7$, and $\|\mathbf{b}\| = 5$.

Norm and Direction -> components

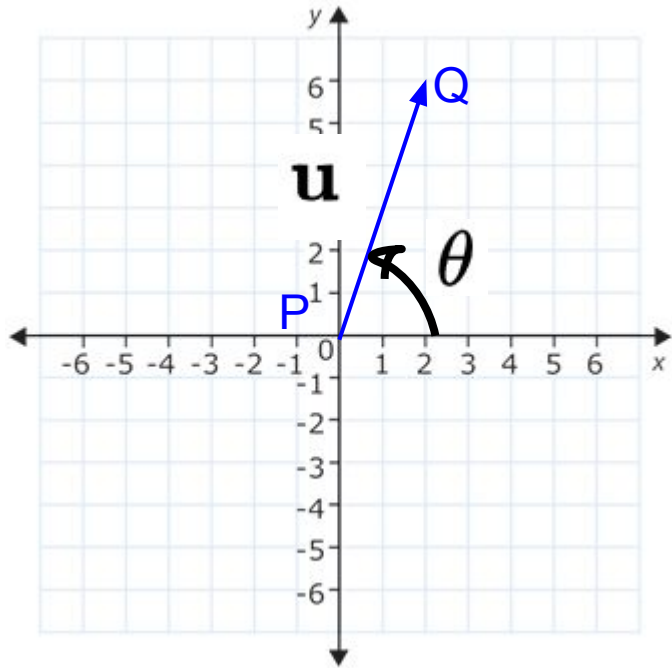
Let the norm (magnitude) of a vector be a and the angle of the vector be θ (theta).

Then, the components of the vector are given as

$$\begin{pmatrix} a \cos \theta \\ a \sin \theta \end{pmatrix}$$



Components -> direction



The angle θ is given by $\theta = \arctan\left(\frac{y}{x}\right)$ or
 $\theta = \pi + \arctan\left(\frac{y}{x}\right)$, depending on the quadrant
that the vector lies in.

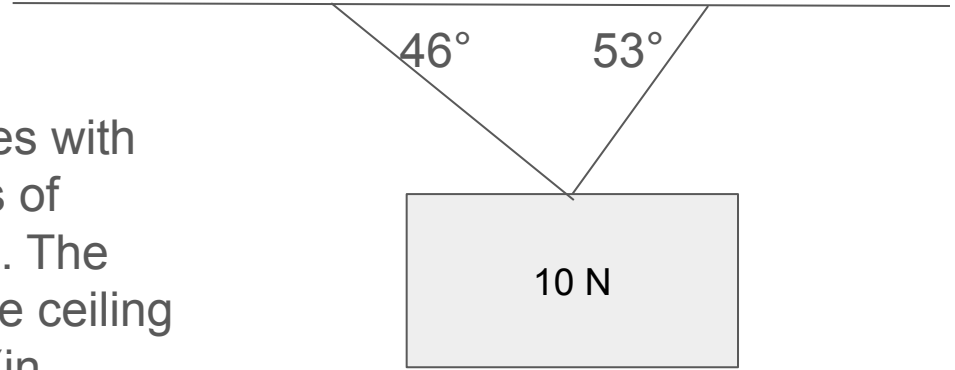
Plane

A plane moves in the direction $N 60^\circ E$ with airspeed (that is, relative to the air around it) 600 mph in the presence of wind blowing from $S 45^\circ E$ with speed 70 mph. Find the actual (relative from the ground) speed of the plane to 3 decimal places and its actual direction in degrees to 3 decimal places.

(Use common sense!)

Forces

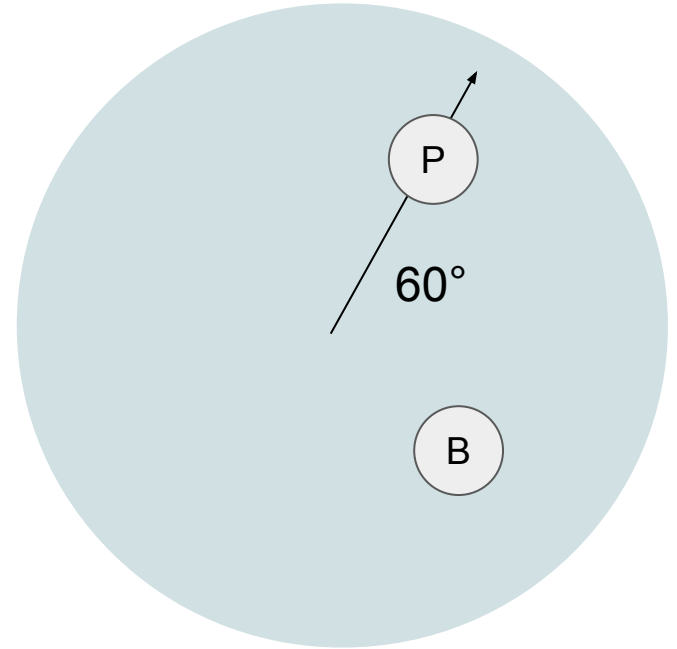
Suppose there is a box of 10 apples with weight 10 N hanged on two pieces of massless string on a stable ceiling. The angles the two strings form with the ceiling are 46° and 53° . Find the tension (in Newtons, to 3 decimal places) in each of the two strings.



Penguins

Barry and Parry are two penguins standing in the middle of an ice lake, with the exact same mass. Barry is sliding at 10 m/s going rightward, and collides with Parry which is initially at rest. If Parry is now moving at 5 m/s at a 30 degree angle east from north, what velocity is Barry now moving at?

Round your answer to three decimal places.



Answers

Plane: 585.798 mph in the direction N 53.372° E

Forces: 6.093 N (53 degree) and 7.033 N (46 degree)

Penguins: 8.660 m/s, 60 degrees east from south (330 deg vector)

Homework for Day 1

#3-60 multiples of 3 on pages 618-619 (PDF pages 705-706)

AoPS challenge problem 9.46: How many vectors in two dimensions have one component equal to 2 and magnitude 6? How are these vectors related geometrically?

Week 5: Vectors

Day 2

Review

1. Evaluate $\left\| \begin{pmatrix} 4 \\ -5 \end{pmatrix} \right\|$.
2. Express $\begin{pmatrix} -1 \\ 6 \end{pmatrix} - 2\begin{pmatrix} -6 \\ 2 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 4 \end{pmatrix}$ as a single vector.
3. Evaluate $\begin{pmatrix} 5 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 4 \end{pmatrix}$.
4. Let $\mathbf{u} = 5\mathbf{i} - 12\mathbf{j}$ and $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$. Find $(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v})$. Also find a unit vector in the direction of $\mathbf{v}$.
5. 2 vectors are orthogonal if they have dot product 0. Which 2 pairs of the following 5 vectors are orthogonal: $\begin{pmatrix} 4 \\ -6 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \end{pmatrix}, \begin{pmatrix} 8 \\ 6 \end{pmatrix}, \begin{pmatrix} 6 \\ 9 \end{pmatrix}, \begin{pmatrix} 9 \\ 6 \end{pmatrix}$.
1 2 3 4 5

Independence

A set of vectors $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ is called independent if the only solution to is $x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n = \vec{0}$ is $x_1 = x_2 = \dots = x_n = 0$.

In two dimensions, any set with one nonzero vector is independent. Also, a set of two vectors **a** and **b** is independent if **a** is not a multiple of **b** and **b** is not a multiple of **a**. A set of three or more vectors is always dependent.

Vectors are called dependent if they are not independent.

A special case is that the zero vector is dependent with any other vector (always 0 times the other vector).

**The zero vector itself is dependent.*

The dot product, revisited

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos \theta$$

Proof: Starting with the law of cosines,

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos(180^\circ - \theta) \\ \Rightarrow (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) &= \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} + 2\|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos \theta \\ \Rightarrow \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} + 2\mathbf{u} \cdot \mathbf{v} &= \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} + 2\|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos \theta \\ \Rightarrow 2\mathbf{u} \cdot \mathbf{v} &= 2\|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos \theta \\ \Rightarrow \boxed{\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos \theta} \end{aligned}$$

Cauchy-Schwarz inequality (four variables, two vectors)

Consider vectors $\langle a, b \rangle$ and $\langle c, d \rangle$. Then because the dot product has at most the magnitude as the product of the magnitudes of the vectors themselves, we have

$$(ac+bd)^2 \leq (a^2+b^2)(c^2+d^2).$$

with equality case if and only if $\langle a, b \rangle$ and $\langle c, d \rangle$ are dependent.

This can be generalized to more dimensions.

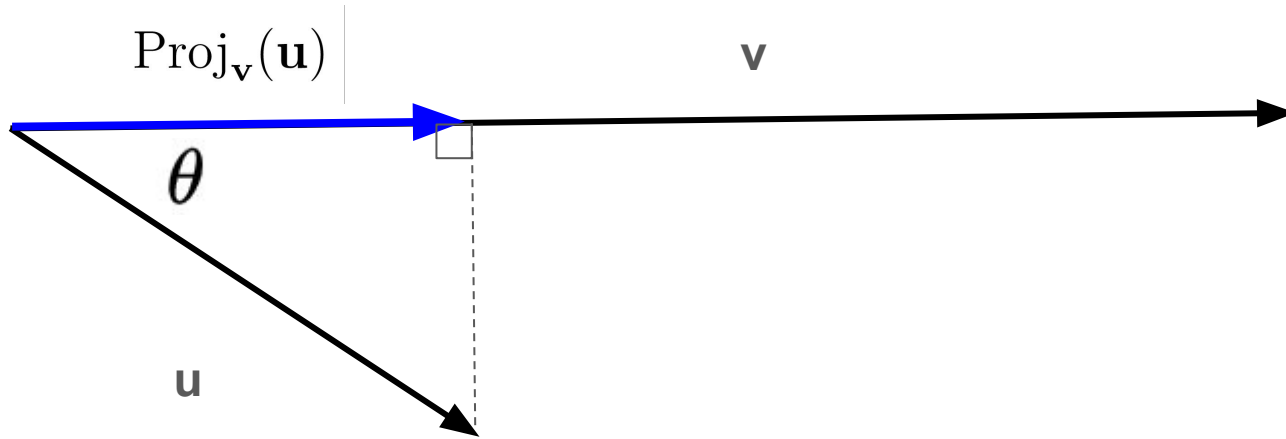
Example Problems

Find the angles between the pairs of vectors.

- (a) $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$
- (b) $\langle 1, 2 \rangle$ and $\langle 3, 4 \rangle$
- (c) $\langle 1, 3 \rangle$ and $\langle 2, 1 \rangle$
- (d) $\langle 2, 5 \rangle$ and $\langle 3, 1 \rangle$
- (e) $\langle 1, 2 \rangle$ and $\langle 3, 6 \rangle$

Definition of Projection

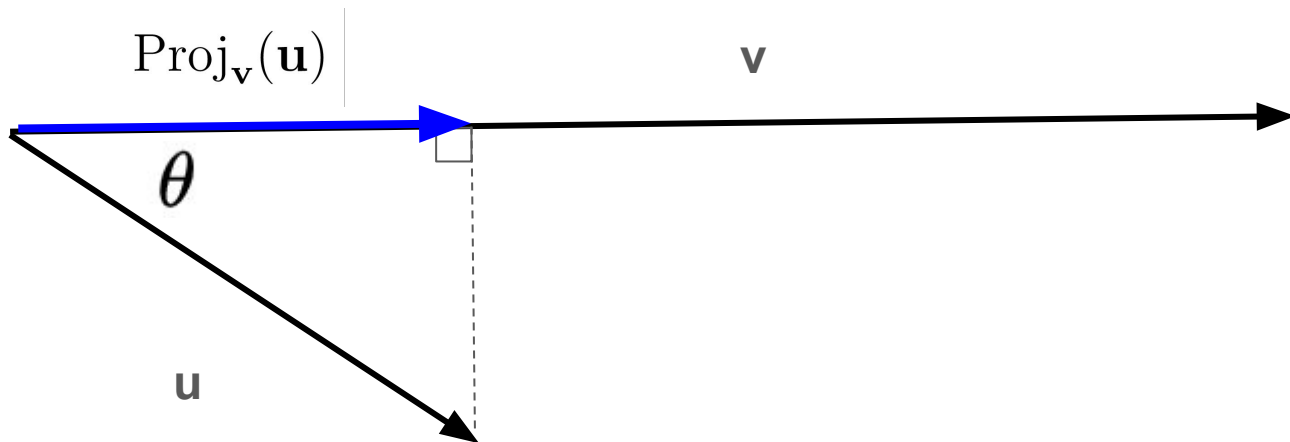
A projection of vector $\mathbf{u}$ onto $\mathbf{v}$ is the vector along $\mathbf{v}$ with starting point as the same starting point $\mathbf{v}$ and endpoint the perpendicular from the endpoint of $\mathbf{u}$ onto $\mathbf{v}$.



Projections

For vectors $\mathbf{u}$ and $\mathbf{v}$ with angle θ , the projection from $\mathbf{u}$ onto $\mathbf{v}$ has length $\|\mathbf{u}\| \cos(\theta)$. We can rewrite this as $\|\mathbf{u}\| \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|}$. The projection will be in the same direction as $\mathbf{v}$, so we multiply by the previous length by the unit vector $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ to get the projection correct. So, the projection from $\mathbf{u}$ onto

$\mathbf{v}$ is $\boxed{\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v}}.$



Finding Projections

Find the projection of (use exact values)

- a. $\langle 0, 1 \rangle$ onto $\langle 2, 2 \rangle$
- b. $\langle 4, 3 \rangle$ onto $\langle 1, 0 \rangle$
- c. $\langle -2, -2 \rangle$ onto $\langle 1, 1 \rangle$
- d. $\langle 1, -4 \rangle$ onto $\langle -3, 2 \rangle$
- e. $\langle 1, 3 \rangle$ onto $\langle -6, -2 \rangle$
- f. $\langle 2, 6 \rangle$ onto $\langle -3, 1 \rangle$

Projections Answers

- a) $\langle 1/2, 1/2 \rangle$
- b) $\langle 4, 0 \rangle$
- c) $\langle -2, -2 \rangle$
- d) $\langle 33/13, -22/13 \rangle$
- e) $\langle 9/5, 3/5 \rangle$
- f) $\langle 0, 0 \rangle$

Problems

Prove the Pythagorean Theorem: $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u} + \mathbf{v}\|^2$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are perpendicular to each other.

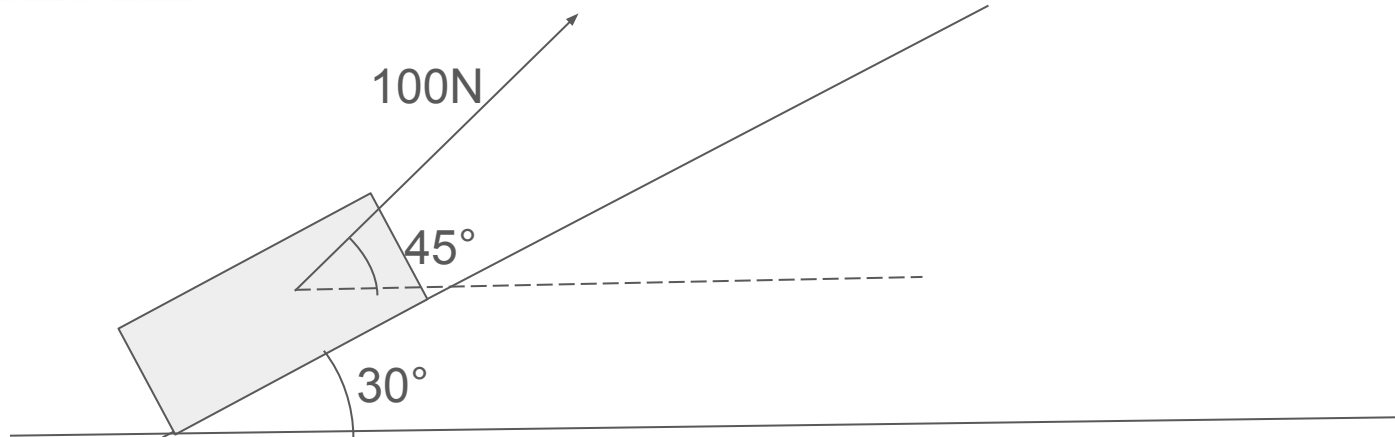
Hints:

- (a) Show that $\mathbf{u} \cdot \mathbf{v} = 0$ implies that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular to each other.
- (b) Expand $\|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v})$.

In physics, work is defined the dot product between the force vector and the distance vector:

$$W = \mathbf{F} \cdot \mathbf{x}$$

- (a) Suppose a force of 100 N that is 45° above the horizontal is applied to a box placed on a frictionless incline with angle 30° above the horizontal. The box moves 50 meters. What is the work done by the force?
- (b) Some people prefer to use the projection of the force onto the incline to solve these problems. What is the projection force, and what is the answer in this case?



Problems

Prove $\|\mathbf{a}\| - \|\mathbf{b}\| \leq \|\mathbf{a} - \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$ using properties of the dot product.

Hint: square both sides

Problem

Let vector $\overrightarrow{AB}$ be $\begin{pmatrix} -x \\ 1 \end{pmatrix}$, vector $\overrightarrow{AC}$ be $\begin{pmatrix} x \\ 2 \end{pmatrix}$, vector $\overrightarrow{AD}$ be $\begin{pmatrix} 2x \\ 0 \end{pmatrix}$.

Suppose $\angle ABC = 180^\circ - \angle ADC$. (Interesting Fact: This means points A,B,C,D lie on a common circle.) If $x > 0$, what is x^2 ?

Hint: This means $\cos(\angle ABC) = -\cos(\angle ADC)$. Use dot products.

Problem

Suppose A, B, C, and O are distinct points such $AO=BO=CO$.

Prove that if $\overrightarrow{OH}=\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}$, segment AH is perpendicular to segment BC.

(Hint: Use the dot product)

Answer

$$\frac{\sqrt{14}}{7}$$

Hard Problems (Not challenge) (skip)

Let $ABCD$ be a parallelogram and P be any point in the same plane. Prove that $PD^2 + PB^2 + AC^2 = AB^2 + AD^2 + PA^2 + PC^2$

Let $ABCD$ be any quadrilateral. Let M be the midpoint of AC and N be the midpoint of BD . Prove that $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4MN^2$.

For interested people, you can use the dot product along with geometric insights with addition and subtraction of vectors.

Homework Day 2

#27-48 multiples of 3 on pages P627-628 (PDF pages 715-716)

Challenge Problem:

Let $\mathbf{a}$ and $\mathbf{b}$ be two nonzero vectors such that $\mathbf{a} + \mathbf{b}$ and $\mathbf{b}$ are orthogonal, and $\mathbf{a} + 2\mathbf{b}$ and $\mathbf{a}$ are orthogonal. Find $\frac{\|\mathbf{a}\|}{\|\mathbf{b}\|}$.

More Challenge problem:

Let v_1, v_2, v_3 be two-dimensional vectors. Prove that v_1, v_2, v_3 are dependent by finding scalars $(c_1, c_2, c_3) \neq (0, 0, 0)$ with $c_1v_1 + c_2v_2 + c_3v_3 = 0$.