

Vectors : Summary

1. Vector addition is commutative. $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$
2. Vector addition is associative. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. Additive identity: $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$.
4. Basic multiplication rules: $0\mathbf{v} = \mathbf{0}$, $1\mathbf{v} = \mathbf{v}$, $-1\mathbf{v} = -\mathbf{v}$
5. Associativity of scalar multiplication: $a(b\mathbf{v}) = (ab)\mathbf{v}$
6. Distributive properties: $(a+b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$, $a(\mathbf{v}+\mathbf{w}) = a\mathbf{v} + a\mathbf{w}$

Review of Vectors!

If $\|\mathbf{v}\| = 4$, what is $\|3\mathbf{v}\|$?

Find a vector $\mathbf{v}$ whose magnitude is 3 and whose component in the $\mathbf{i}$ direction is equal to the component in the $\mathbf{j}$ direction.

Review

Find the angle and dot product between the following pairs of vectors :

$\langle 6, 7 \rangle$ and $\langle 4, 2 \rangle$

$\langle 6, 7 \rangle$ and $\langle -6, -7 \rangle$

Find the projections below :

$\langle 6, 7 \rangle$ onto $\langle 4, 2 \rangle$

$\langle 1, 3 \rangle$ onto $\langle 369369, 123123 \rangle$

Review : Physics

During cruise, a plane flies at an airspeed 960 km/h at a heading of 315 degrees. A headwind blows into the plane at 50 km/h at a heading of 150 degrees. Find the actual speed of the plane to 3 decimal places.

Review : Work

Find the work done by a force of 50N 30 degrees above the horizontal that moved a 30kg box 60m up an incline of 60 degrees.

Find the projection of the force onto the incline.

What is a matrix (Low cortisol)?



rectangular array of numbers

Examples:

$$\begin{pmatrix} 6 & 7 \\ 7 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 2 & 3 \\ 1 & 6 & 2 \end{pmatrix}$$

We'll only be looking at 2x2 matrices today, which are of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

What is a matrix (High cortisol)?



a **function** that takes a vector and transforms it into another vector.

Explicitly, for 2×2 matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

For matrix A and vectors x and y :

$$A(x + y) = Ax + Ay$$

Very special examples



Zero matrix:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

sends all vectors to the zero vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Identity matrix:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

sends all vectors to themselves

Multiplication by scalars



Given matrix A represented by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and scalar (aka real number) k , we have

$$k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$

Important property:

$$k(Av) = A(kv) = k(Av)$$

Addition



Add corresponding entries:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

For matrices A , B and vector v :

$$(A + B)v = Av + Bv$$

It's commutative and associative!

Multiplication



Since matrices are functions, we can compose them to form new functions. Given matrices A and B , we can form the matrix AB which is the function that sends a vector v to $A(Bv)$.

For 2×2 matrices, we have:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

Its associative and distributive, but NOT COMMUTATIVE!

Example : Multiplication

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -6 & 6 & 1 & 0 \\ 2 & 5 & 4 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ -3 & 2 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 2 & 8 & -1 \\ 3 & 6 & 0 \end{bmatrix}$$

Determinant (some dark magic)



Given a matrix A represented by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we define the determinant of A to be $ad - bc$.

We write determinant as $\det(A)$ or $|A|$.

By copious amounts of dark magic, $\det(AB) = \det(A) \det(B)$

Examples : Find determinants

$$\begin{vmatrix} 6 & 4 \\ -1 & 3 \end{vmatrix}$$

$$\begin{vmatrix} 8 & -3 \\ 4 & 2 \end{vmatrix}$$

More Determinant Equations

$$\begin{vmatrix} x & 1 \\ 3 & x \end{vmatrix} = -2$$

Cramer's rule (more dark magic)



Given system of equations

$$ax + by = u$$

$$cx + dy = v$$

by an excessively bountiful amount of dark magic, the solution is given by

$$x = \frac{\begin{vmatrix} u & b \\ v & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

,

$$y = \frac{\begin{vmatrix} a & u \\ c & v \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

Examples : Using Cramer's Rule

$$\begin{cases} 4x + 5y = -3 \\ -2y = -4 \end{cases}$$

$$\begin{cases} 2x + 4y = 16 \\ 3x - 5y = -9 \end{cases}$$

$$\begin{cases} -x + 2y = 5 \\ 4x - 8y = 6 \end{cases}$$



Example

Let $\mathbf{T} \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ and $\mathbf{T} \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$. What is $\mathbf{T} \begin{pmatrix} 4 & -3 \\ -3 & 1 \end{pmatrix}$?

Challenge Example

Let $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

(a) Prove that $\mathbf{A}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$, where F_n denotes the n^{th} Fibonacci number.

Follow Up Challenge

(b) Prove that $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ for all $n \geq 0$.

Find the determinant

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

Homework

Standard Problems - *Precalculus Enhanced with Graphing Utilities*

Section 10.3, Pages 767-768, Problems 10-12, 29-33, 47, 51-56, 58, 59

*Ignore parts of questions which involve graphing utilities and do all calculations by hand.

Challenge Problems - AoPS *Precalculus*

Page 362 Problem 10.4.4

Page 377, Problem 10.64

End of Class 1

Example

A 2×2 matrix is **diagonal** if it is of the form $\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$. Show that the product of two diagonal matrices is also diagonal.

Example

Let $\mathbf{C}$ be the matrix formed by doubling the entries in the first row of $\mathbf{A}$. How are $\det(\mathbf{A})$ and $\det(\mathbf{C})$ related?

Example

The first column of $\mathbf{B}$ is the same as the first column of $\mathbf{A}$. The second column of $\mathbf{B}$ is the sum of the second column of $\mathbf{A}$ and k times the first column of $\mathbf{A}$. Show that $\det(\mathbf{A}) = \det(\mathbf{B})$.

Inverse



Given matrix A , the inverse of A is the matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$ where I is the identity matrix.

For 2×2 matrices, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\det(A) \neq 0$, then $A^{-1} = \left(\frac{1}{\det(A)}\right) \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Example : Finding Inverses

$$\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 15 & 3 \\ 10 & 2 \end{bmatrix}$$

Bonus dark magic:



If $B = kA$, then

$$\det(B) = k^2 \det(A)$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a+kc & b+kd \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c+ka & d+kb \end{vmatrix} = \begin{vmatrix} a+kb & b \\ c+kd & d \end{vmatrix} = \begin{vmatrix} a & b+kd \\ c & d+kc \end{vmatrix}$$

Examples : Solving Matrices with Inverses

$$2x + y = 8$$

$$x + y = 5$$

$$3x - y = 8$$

$$-2x + y = 4$$

Ellipses

ELLIPSE WITH CENTER AT THE ORIGIN

The graph of each of the following equations is an ellipse with center at the origin and having the given properties.

EQUATION $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$a > b > 0$$

VERTICES

$$(\pm a, 0)$$

MAJOR AXIS

Horizontal, length $2a$

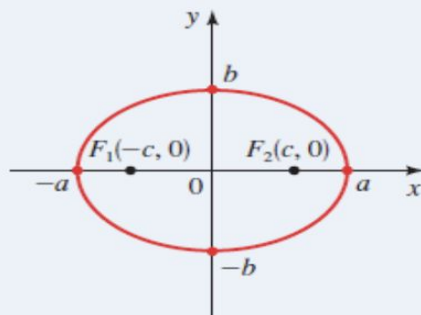
MINOR AXIS

Vertical, length $2b$

FOCI

$$(\pm c, 0), c^2 = a^2 - b^2$$

GRAPH



$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

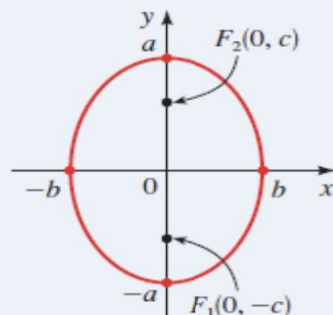
$$a > b > 0$$

$$(0, \pm a)$$

Vertical, length $2a$

Horizontal, length $2b$

$$(0, \pm c), c^2 = a^2 - b^2$$



Review : Ellipses

Remember that the equation of an ellipse is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ for an ellipse centered at (h, k).

Find the foci points of the ellipse in terms of h, k, a, and b.

Ellipse Review Problems

An ellipse has foci at $(0, 0)$ and $(14, 0)$ and passes through the vertex of the parabola with equation $y = x^2 - 10x + 37$. Find the length of the major axis of this ellipse.

Review Example

Find the area enclosed in the graph of $x^2 + y^2 = 16x + 32y$.

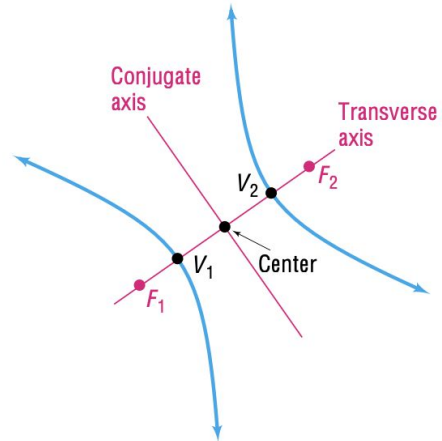
Hyperbolas : Introduction

A hyperbola is a set of points whose absolute difference in distance to two points, which are known as the **foci**, are equal.

The **transverse axis** is the line joining the foci, its midpoint is the center.

The **conjugate axis** is the line perpendicular to the transverse axis passing through the center.

The two separate curves are called **branches**.



Equation and Asymptotes of Hyperbolas

The equation of a hyperbola is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

for a hyperbola centered at (h, k) , and the **asymptotes** are $y - k = \pm \frac{b}{a}(x - h)$

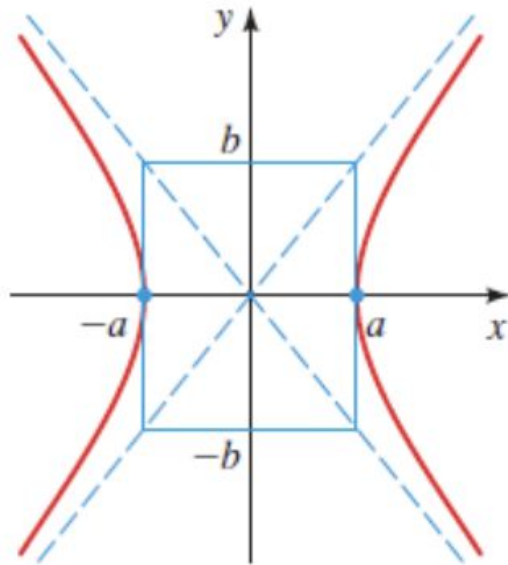
Derivation of asymptotes :

$$y = k \pm b \sqrt{\frac{(x-h)^2}{a^2} - 1}$$

Taking limits of both sides,

$$\lim_{x \rightarrow \pm\infty} y = \lim_{x \rightarrow \pm\infty} \left[k \pm \frac{b}{a}(x-h) \sqrt{1 - \frac{a^2}{(x-h)^2}} \right]$$

$$y = k \pm \frac{b}{a}(x-h)$$



Derivations

Equation : Using the definition of a hyperbola and the distance formula, we have

$$\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = \pm 2a$$

where the 2 foci are (0, c) and (0, -c)

$$\sqrt{(x + c)^2 + y^2} = \pm 2a + \sqrt{(x - c)^2 + y^2}$$

$$(x + c)^2 + y^2 = 4a^2 \pm 4a\sqrt{(x - c)^2 + y^2} + (x - c)^2 + y^2$$

after rearranging and squaring

$$cx - a^2 = \pm a\sqrt{(x - c)^2 + y^2}$$

after expanding and cancelling out terms

$$(cx - a^2)^2 = a^2[(x - c)^2 + y^2]$$

$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2)$$

after squaring and cancelling

Now, let $b^2 = c^2 - a^2$ and we get

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Hyperbolas on the y-axis

For a hyperbola centered at (h, k) , the equation takes the form $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

Its asymptotes are the lines $y - k = \pm \frac{a}{b} (x - h)$

Summary

HYPERBOLA WITH CENTER AT THE ORIGIN

The graph of each of the following equations is a hyperbola with center at the origin and having the given properties.

EQUATION $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (a > 0, b > 0)$

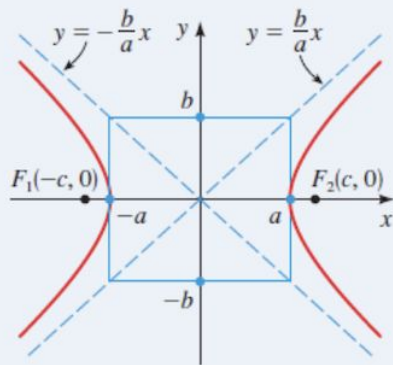
VERTICES $(\pm a, 0)$

TRANSVERSE AXIS Horizontal, length $2a$

ASYMPTOTES $y = \pm \frac{b}{a}x$

FOCI $(\pm c, 0), \quad c^2 = a^2 + b^2$

GRAPH



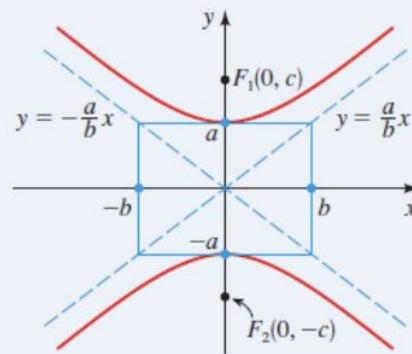
$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad (a > 0, b > 0)$

$(0, \pm a)$

Vertical, length $2a$

$y = \pm \frac{a}{b}x$

$(0, \pm c), \quad c^2 = a^2 + b^2$



Example **Discuss the equation:** $\frac{x^2}{16} - \frac{y^2}{4} = 1$

Find the asymptotes, foci, and length of the transverse and conjugate axii.

Example

Find the center, Asymptotes, vertices and foci of the following graph

$$\frac{(x - 3)^2}{9} - y^2 = 1$$

Example

$$x^2 + 12x - 3y^2 + 18y + 21 = 0$$

Examples : Find the equation

Center at $(0, 0)$; focus at $(3, 0)$; vertex at $(1, 0)$

Example

Foci at $(-5, 0)$ and $(5, 0)$; vertex at $(3, 0)$

Example

Foci at $(-4, 0)$ and $(4, 0)$; asymptote the line $y = -x$

Real World Example

Suppose that two people standing 1 mile apart both see a flash of lightning. After a period of time, the person standing at point A hears the thunder. One second later, the person standing at point B hears the thunder. If the person at B is due west of the person at A and the lightning strike is known to occur due north of the person standing at point A, where did the lightning strike? Use 1100 feet per second as the speed of sound.

Challenging Example

The graph of the equation $\frac{(x-3)^2}{a^2} - \frac{y^2}{b^2} = 1$, one of the points of intersection of the hyperbola and the line $y = ax$ has x-coordinate 0. Find the x-coordinate of the points of intersection of the line $y = bx$ and hyperbola.

Homework

Standard Problems - *Precalculus Enhanced with Graphing Utilities*

Section 10.4, Pages 783-785, Problems 7, 11, 15, 17-23, 31-33, 44-46, 59, 75

*Ignore parts of questions which involve graphing utilities and do all calculations by hand.

Challenge Problems - AoPS *Precalculus*

Page 378-379, Problems 10.75 and 10.76

Homework

Standard Problems - *Precalculus Enhanced with Graphing Utilities*

Section 9.4, Problems 19, 31, 41, 47